

Compact exact Lagrangian intersection in cotangent bundles via sheaf quantization

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Abstract

We prove that the cardinality of the intersection of two compact exact Lagrangian submanifolds in a cotangent bundle is bounded from below by the dimension of the Hom space of the Guillermou's sheaf quantizations of the Lagrangians in Tamarkin's category. This gives a purely sheaf-theoretic new proof of a result of Nadler and Fukaya-Seidel-Smith, which asserts that the cardinality is at least the sum of the Betti numbers of the base space.

1 Introduction

An important problem in symplectic geometry is to study exact Lagrangian submanifolds in cotangent bundles. Arnold's nearby Lagrangian conjecture asserts that any compact exact Lagrangian submanifold of a cotangent bundle is Hamiltonian isotopic to the zero-section. In symplectic geometry, non-displaceability of Lagrangian submanifolds is also a central problem, which is also originally due to Arnold. Recall that a diffeomorphism $\psi: X \rightarrow X$ of a symplectic manifold X is said to be a Hamiltonian diffeomorphism if there exists a Hamiltonian isotopy with compact support $\phi = (\phi_s)_{s \in [0,1]}: X \times [0,1] \rightarrow X$ such that $\phi_1 = \psi$ and $\phi_0 = \text{id}_X$. The problem is the existence or non-existence of a Hamiltonian diffeomorphism ψ satisfying $L_1 \cap \psi(L_2) = \emptyset$ for two compact Lagrangians L_1 and L_2 and the estimation of the cardinality of the intersection. For a cotangent bundle and its zero-section, Laudenchbach-Sikorav [LS85] proved the following. Let M be a compact C^∞ -manifold without boundary and T^*M its cotangent bundle equipped with the canonical exact symplectic structure. Then for a Hamiltonian diffeomorphism ψ of T^*M such that the intersection of the zero-section M and its image $\psi(M)$ is transversal, one has an inequality

$$\#(M \cap \psi(M)) \geq \sum_{l \in \mathbb{Z}} \dim H^l(M; \mathbb{Q}). \quad (1.1)$$

Note that this inequality had been proved by Chaperon [Cha83] in the case M is a torus.

Although Arnold's nearby Lagrangian conjecture has been proved only for some special cases, combined with the inequality (1.1), it implies the following statement for any two compact exact Lagrangians. In fact, this was already proved by Nadler [Nad09] and Fukaya-Seidel-Smith [FSS08] with the use of the Floer theory.

Theorem 1.1 ([Nad09, Theorem 1.3.1] and [FSS08, Theorem 1]). *For any two compact connected exact Lagrangian submanifolds L_1 and L_2 of T^*M intersecting transversally, one has*

$$\#(L_1 \cap L_2) \geq \sum_{l \in \mathbb{Z}} \dim H^l(M; \mathbb{Q}). \quad (1.2)$$

In this paper, we study intersection of two compact exact Lagrangian submanifolds in a cotangent bundle using a different method based on the microlocal sheaf theory, especially Guillermou's sheaf quantization and the category considered by Tamarkin. In particular, we give a purely sheaf-theoretic new proof of Theorem 1.1.

The microlocal sheaf theory was introduced and systematically developed by Kashiwara and Schapira [KS85, KS90]. One of the key ingredients of the theory is the notion of microsupports of sheaves. Let us recall it briefly. In the sequel, let k be a field. For a C^∞ -manifold X , we denote by $\mathbf{D}^b(X)$ the bounded derived category of sheaves of k -vector spaces. With an object $F \in \mathbf{D}^b(X)$, one can associate its microsupport $\mathrm{SS}(F)$, a closed subset of the cotangent bundle T^*X of X . Roughly speaking, $\mathrm{SS}(F)$ describes directions in which the cohomology of F does not propagate. The microsupport is conic, that is, invariant by the action of $\mathbb{R}_{>0}$ on T^*X .

Tamarkin [Tam08] gave a new approach to non-displaceability problems based on the microlocal sheaf theory. Roughly speaking, he constructed a sheaf whose microsupport coincides with a given Lagrangian submanifold of a cotangent bundle outside the zero-section and deduced a non-displaceability result from it. Such a sheaf is called a *sheaf quantization* of the Lagrangian. However, the problem is that microsupports of sheaves are conic and there is no such sheaf for a non-conic Lagrangian. Tamarkin's idea to treat non-conic Lagrangians and not necessarily homogeneous Hamiltonian isotopies is to add a variable and reduce the problem to the exact cases. One can consider a sheaf quantization for a non-conic Lagrangian by adding a variable and "conifying" it. After Tamarkin's work, Guillermou-Kashiwara-Schapira [GKS12] and Guillermou [Gui12, Gui16] proved the existence of sheaf quantization of Hamiltonian isotopies and compact exact Lagrangian submanifolds in cotangent bundles. Below we review their results more precisely, following the formulation of Guillermou-Schapira [GS14] for the result of Tamarkin [Tam08]. See Section 3 for more details. Note that sheaf-theoretic approaches to symplectic geometry also appeared in [KO01, NZ09, Nad09].

In what follows, again let M be a non-empty compact connected C^∞ -manifold without boundary and regard its cotangent bundle T^*M as an exact symplectic manifold.

Tamarkin [Tam08] considered a category $\mathcal{D}(M)$ defined by a localization of $\mathbf{D}^b(M \times \mathbb{R})$ by objects whose microsupports are contained in $\{\tau \leq 0\} \subset T^*(M \times \mathbb{R})$, where $(t; \tau)$ is the homogeneous symplectic coordinate of $T^*\mathbb{R}$. For a compact subset A of T^*M , one defines a subcategory $\mathcal{D}_A(M)$ by the subcategory consisting of F with $\mathrm{SS}(F) \cap \{\tau > 0\} \subset \rho^{-1}(A)$, where $\rho: \{\tau > 0\} \rightarrow T^*M, (x, t; \xi, \tau) \mapsto (x; \xi/\tau)$. Then one can prove that if $A \cap B = \emptyset$, $\mathrm{Hom}_{\mathcal{D}(M)}(F, G) \simeq 0$ for any $F \in \mathcal{D}_A(M)$ and $G \in \mathcal{D}_B(M)$. Moreover we define a category $\mathcal{T}(M)$ as a quotient of $\mathcal{D}(M)$ by what are called torsion objects. The category $\mathcal{T}(M)$ has a Hamiltonian invariance property. Namely, for a Hamiltonian diffeomorphism $\psi: T^*M \rightarrow T^*M$, one can construct a functor $\Psi: \mathcal{D}_A(M) \rightarrow \mathcal{D}_{\psi(A)}(M)$ such that $\Psi(F) \simeq F$ in $\mathcal{T}(M)$ for any $F \in \mathcal{D}_A(M)$. The fact that Hom spaces in $\mathcal{T}(M)$ can be described as inductive limits of those in $\mathcal{D}(M)$ implies the following non-displaceability theorem: if A and B are compact subsets of T^*M and if there exist objects $F \in \mathcal{D}_A(M)$ and $G \in \mathcal{D}_B(M)$ such that $\mathrm{Hom}_{\mathcal{T}(M)}(F, G) \neq 0$, then A and B are mutually non-displaceable, that is, $A \cap \psi(B) \neq \emptyset$ for any Hamiltonian diffeomorphism ψ of T^*M .

After the work of Tamarkin, Guillermou-Kashiwara-Schapira [GKS12] constructed a sheaf quantization of a Hamiltonian isotopy of T^*M and proved the non-displaceability of the zero-section of a cotangent bundle by using the quantization. Let I be an open interval of \mathbb{R} containing 0 and $\phi = (\phi_s)_{s \in I}: T^*M \times I \rightarrow T^*M$ a Hamiltonian isotopy such that $\phi_0 = \mathrm{id}_{T^*M}$. Adding a variable, one can canonically associate a homogeneous Hamiltonian isotopy $\hat{\phi}: \hat{T}^*(M \times \mathbb{R}) \times I \rightarrow \hat{T}^*(M \times \mathbb{R})$, where $\hat{T}^*(M \times \mathbb{R})$ is the complement of the zero-

section. Denote by $\Lambda_{\widehat{\phi}} \subset \mathring{T}^*(M \times \mathbb{R}) \times \mathring{T}^*(M \times \mathbb{R}) \times T^*I$ the conic Lagrangian submanifold associated with $\widehat{\phi}$. The authors proved the existence and uniqueness of a sheaf quantization $K \in \mathbf{D}(M \times \mathbb{R} \times M \times \mathbb{R} \times I)$ of $\Lambda_{\widehat{\phi}}$ whose restriction to $s = 0$ is the constant sheaf of the diagonal. Using the quantization, one can construct a family of sheaves $(F_s)_{s \in I}$ such that the microsupport of F_s is related with the intersection $M \cap \phi_s(M)$ of the zero-section and its image by ϕ_s and all F_s have the same cohomology. The authors recovered the result of Laudenbach-Sikorav [LS85] by applying the microlocal Morse lemma and the Morse inequality for sheaves to F_s . Note also that the existence of the functor Ψ mentioned above is now a consequence of the existence of K .

After these works, Guillermou [Gui12, Gui16] constructed a sheaf quantization of a compact exact Lagrangian submanifold of a cotangent bundle and studied topological properties of the Lagrangian by using the quantization. Let L be a compact connected exact Lagrangian submanifold of T^*M . Choosing a primitive of the Liouville 1-form, one can define a conification \widehat{L} , which is a conic Lagrangian submanifold of $T^*(M \times \mathbb{R})$. Guillermou proved the existence and uniqueness of a sheaf quantization $F_{\widehat{L}} \in \mathbf{D}^b(M \times \mathbb{R})$ of \widehat{L} such that $F_{\widehat{L}}|_{M \times \{-t\}}$ is 0 and $F_{\widehat{L}}|_{M \times \{t\}}$ is isomorphic to the constant sheaf on M for a sufficiently large $t > 0$. The object $F_{\widehat{L}}$ is called the canonical sheaf quantization. Using the quantization, he recovered results of Abouzaid and Kragh [Kra13], saying that the Maslov class of L is zero, and Fukaya-Seidel-Smith [FSS09] and Abouzaid [Abo12], saying that the projection $L \rightarrow M$ is a homotopy equivalence.

In this paper, we study the relation between intersection of two compact exact Lagrangian submanifolds of a cotangent bundle T^*M and Hom spaces of Guillermou's sheaf quantizations in the category $\mathcal{T}(M)$. Let L_1 and L_2 be compact connected exact Lagrangian submanifolds of T^*M and chose primitives $f_i: L_i \rightarrow \mathbb{R}$ of the Liouville 1-form. Let us denote by $F_i \in \mathbf{D}^b(M \times \mathbb{R})$ the canonical sheaf quantization associated with (L_i, f_i) .

Proposition 1.2. *One has an isomorphism*

$$\mathrm{Hom}_{\mathcal{T}(M)}(F_1, F_2[l]) \simeq H^l(M; k_M) \quad \text{for any } l \in \mathbb{Z}. \quad (1.3)$$

In particular, this proposition implies that $L_1 \cap L_2 \neq \emptyset$ for any compact connected exact Lagrangian submanifolds L_1 and L_2 by Tamarkin's non-displaceability theorem. Note that for the special case where one of the Lagrangian is the zero-section, the non-emptiness was proved by Gromov [Gro85].

Moreover we prove that the Hom space can be used not only for the non-displaceability but also for the estimate of the cardinality of the intersection. Namely, we prove that the cardinality is bounded from below by the dimension of the Hom space.

Theorem 1.3. *Assume that L_1 and L_2 intersect transversally. Then one has an inequality*

$$\#(L_1 \cap L_2) \geq \sum_{l \in \mathbb{Z}} \dim \mathrm{Hom}_{\mathcal{T}(M)}(F_1, F_2[l]). \quad (1.4)$$

This theorem is proved by a combination of techniques of the above three works. By results of Tamarkin [Tam08], one can construct a sheaf $\mathcal{H}om^*(F_1, F_2)$ on $M \times \mathbb{R}$ whose l -th cohomology (supported by $M \times [0, +\infty)$) gives $\mathrm{Hom}_{\mathcal{D}(M)}(F_1, F_2[l])$. Then we apply the Morse inequality for sheaves to $\mathcal{H}om^*(F_1, F_2)$ and the function $t: M \times \mathbb{R} \rightarrow \mathbb{R}, (x, t) \mapsto t$ as in Guillermou-Kashiwara-Schapira [GKS12] and take an inductive limit. The key proposition is that the dimension of the stalk of the local cohomology sheaf $\bigoplus_{l \in \mathbb{Z}} H^l_{M \times [t_0, +\infty)}(\mathcal{H}om^*(F_1, F_2))_{(x_0, t_0)}$ is equal to the cardinality of $\{p \in L_1 \cap L_2 \mid x_0 = \pi(p), t_0 = f_1(p) - f_2(p)\}$, where $\pi: T^*M \rightarrow M$ is the projection. In order to prove

this, we use microlocal inverse images and some techniques developed by Kashiwara-Schapira [KS85, KS90]. Combining Proposition 1.2 with Theorem 1.3, we can recover Theorem 1.1 as a corollary. In fact, using sections of $\mathcal{H}om^*(F_1, F_2)$, we can obtain more precise estimates including the difference of the values of the primitives $f_1(p) - f_2(p)$ (see Proposition 4.9 for more details).

By the result of Tamarkin [Tam08], Guillermou [Gui12, Gui16], and ours, the category $\mathcal{T}(M)$ has the following three properties: (i) Hamiltonian invariance ([Tam08]), (ii) $\mathrm{Hom}_{\mathcal{T}(M)}(F_{\widehat{L}}, F_{\widehat{L}}[l]) \simeq H^l(L; k)$ ($l \in \mathbb{Z}$) for a compact exact Lagrangian L (a particular case of Proposition 1.2 and by results in [Gui12, Gui16]), and (iii) the dimension of the Hom space is a lower bound of the cardinality of the intersection (Theorem 1.3). The Floer cohomology $HF^*(L_1, L_2)$ has similar properties to (i)–(iii), though the approach is totally different. In the Floer theory, one defines the Floer cohomology so that its rank gives a lower bound of the intersection, but the proof of Hamiltonian invariance is difficult. In contrast, in the sheaf-theoretic approach, one constructs the category $\mathcal{T}(M)$ so that it satisfies the property (i), but (iii) is unclear from the definition.

Although our approach is purely sheaf-theoretic and does not use Floer cohomology or Fukaya categories, it seems to be closely related to approaches by Nadler [Nad09] and Fukaya-Seidel-Smith [FSS08, FSS09]. In order to discuss the relation, let us review their results briefly. Nadler [Nad09] proved the following after the work of Nadler-Zaslow [NZ09]. Let Z be a compact real analytic manifold without boundary. Then the infinitesimal Fukaya category $\mathfrak{Fuk}(T^*Z)$ is equivalent to the bounded derived category of constructible sheaves $\mathbf{D}_c^b(Z)$. Moreover, assuming that Z is simply-connected, he proved any relatively spin compact connected exact Lagrangian submanifold L with vanishing Maslov class of T^*Z corresponds to (a shift of) the constant sheaf of rank 1 on Z . In particular, these imply that L is isomorphic to (a shift of) the zero-section Z in $\mathfrak{Fuk}(T^*Z)$ and for any pair of such Lagrangian submanifolds (L_1, L_2) , one has an isomorphism (up to shift)

$$HF^*(L_1, L_2) \simeq H^*(Z). \quad (1.5)$$

By the definition of the Floer cohomology, the isomorphism (1.5) implies Theorem 1.1. Fukaya-Seidel-Smith [FSS08, FSS09] independently proved the isomorphism in $\mathfrak{Fuk}(T^*Z)$ and (1.5) for a C^∞ -manifold Z (not necessarily simply-connected, see [FSS09, Theorem 1.3]). Note that their assumptions of relatively spin and vanishing Maslov class can be removed, thanks to results of Abouzaid [Abo12], and Abouzaid and Kragh [Kra13], respectively.

Also in our approach, we can prove an isomorphism between any compact exact Lagrangian and the zero-section in terms of sheaf quantization. In fact, as pointed out by T. Kuwagaki, for any compact connected exact Lagrangian submanifold L , the canonical sheaf quantization $F_{\widehat{L}}$ is isomorphic to the canonical sheaf quantization $F_{\widehat{M}}$ associated with the zero-section M of T^*M in the category $\mathcal{T}(M)$ (see Proposition 4.4). In our ongoing work with him, we extend the notion of sheaf quantization to non-compact Lagrangians and explore a relation between the Nadler-Zaslow equivalence and Guillermou's sheaf quantization. During the preparation of this paper, the author also learned that C. Viterbo announced that he had proved the relation between the section of $\mathcal{H}om^*(F_1, F_2)$ and the Floer cochain complex $CF(L_1, L_2)$ ¹.

This paper is organized as follows. In Section 2, we recall the microlocal sheaf theory due to Kashiwara and Schapira [KS85, KS90]. In Section 3, we review the results of [Tam08, GKS12, GS14] about Tamarkin's non-displaceability theorem and sheaf quantization of Hamiltonian isotopies and compact exact Lagrangian submanifolds in cotangent

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bundles. In Section 4, we prove Proposition 1.2 and Theorem 1.3. In Appendix, we prove the “functoriality” of Guillermou’s canonical sheaf quantization with respect to Hamiltonian diffeomorphisms, which seems to be useful for further developments. Note that results in Appendix are independent and not used in the previous sections.

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2 Preliminaries on microlocal sheaf theory

In this paper, all manifolds are assumed to be real manifolds of class C^∞ . Throughout this paper, let k be a field.

In this section, we recall some definitions and results from [KS85, KS90]. We mainly follow the notation in [KS90].

2.1 Geometric notions ([KS90, §4.3, §A.2])

Let X be a C^∞ -manifold of dimension d_X . For a locally closed subset A of X , we denote by \overline{A} its closure. One also denotes by Δ_X or simply Δ the diagonal of $X \times X$. We denote by $\tau_X: TX \rightarrow X$ the tangent bundle of X and by $\pi_X: T^*X \rightarrow X$ the cotangent bundle of X . If there is no risk of confusion, we simply write π instead of π_X . If X is a submanifold of a manifold Y , one denotes by T_X^*Y the conormal bundle to X in Y . In particular, T_X^*X denotes the zero-section of T^*X . We set $\dot{T}^*X := T^*X \setminus T_X^*X$.

Let $f: X \rightarrow Y$ be a morphism of manifolds. With f we associate morphisms and a commutative diagram

$$\begin{array}{ccccc} T^*X & \xleftarrow{f_d} & X \times_Y T^*Y & \xrightarrow{f_\pi} & T^*Y \\ \downarrow \pi_X & & \downarrow \pi & & \downarrow \pi_Y \\ X & \xlongequal{\quad} & X & \xrightarrow{f} & Y, \end{array} \quad (2.1)$$

where f_π is the projection and f_d is induced by the transpose of the tangent map $f': TX \rightarrow X \times_Y TY$. We also set

$$T_X^*Y := \text{Ker } f_d = f_d^{-1}(T_X^*X) \subset X \times_Y T^*Y. \quad (2.2)$$

If f is a closed embedding, the above notation coincides with the previous one.

We denote by $(x; \xi)$ a local homogeneous coordinate system of T^*X . The cotangent bundle T^*X is an exact symplectic manifold with the Liouville form $\alpha = \langle \xi, dx \rangle$. The antipodal map $a: T^*X \rightarrow T^*X$ is defined by $(x; \xi) \mapsto (x; -\xi)$. For a subset A of T^*X , we denote by A^a its image by the map a .

2.2 Microsupports of sheaves ([KS90, §5.1])

We denote by k_X the constant sheaf with stalk k and by $\mathbf{D}^b(X) = \mathbf{D}^b(k_X)$ the bounded derived category of sheaves of k -vector spaces on X . For a morphism of manifolds $f: X \rightarrow Y$, one can define Grothendieck's six operations Rf_* , f^{-1} , $Rf_!$, $f^!$, $R\mathcal{H}om$, \otimes between derived categories of sheaves. Since we work on the field k , we simply write \otimes instead of $\overset{L}{\otimes}$. For an inclusion $i_Z: Z \rightarrow X$ of a locally closed subset Z of X and $F \in \mathbf{D}^b(X)$, we define

$$F_Z := i_{Z!} i_Z^{-1} F, \quad R\Gamma_Z(F) := Ri_{Z*} i_Z^! F. \quad (2.3)$$

We simply write k_Z for $(k_X)_Z$. One denotes by $\omega_X \in \mathbf{D}^b(X)$ the dualizing complex on X , that is, $\omega_X := a_X^! k$, where $a_X: X \rightarrow \text{pt}$ is the natural morphism. Note that ω_X is isomorphic to $\text{or}_X[d_X]$, where or_X is the orientation sheaf on X . More generally, for a morphism of manifolds $f: X \rightarrow Y$, we denote by $\omega_f = \omega_{X/Y} := f^! k_Y \simeq \omega_X \otimes f^{-1} \omega_Y^{\otimes -1}$ the relative dualizing complex.

Recall the definition of the microsupport $\text{SS}(F)$ of $F \in \mathbf{D}^b(X)$.

Definition 2.1 ([KS90, Definition 5.1.2]). Let $F \in \mathbf{D}^b(X)$ and $p \in T^*X$. One says that $p \notin \text{SS}(F)$ if there is a neighborhood U of p in T^*X such that for any x_0 and any C^∞ -function φ on X (defined on a neighborhood of x_0) with $d\varphi(x_0) \in U$, one has $R\Gamma_{\{\varphi \geq \varphi(x_0)\}}(F)_{x_0} \simeq 0$.

The following properties can be checked from the definition of microsupports.

- (i) The microsupport is a conic (i.e., invariant by the action of $\mathbb{R}_{>0}$ on T^*X) closed subset of T^*X .
- (ii) $\text{SS}(F) \cap T_X^*X = \pi(\text{SS}(F)) = \text{Supp}(F)$.
- (iii) The microsupports satisfy the triangular inequality: if $F_1 \rightarrow F_2 \rightarrow F_3 \xrightarrow{+1}$ is a distinguished triangle in $\mathbf{D}^b(X)$, $\text{SS}(F_i) \subset \text{SS}(F_j) \cup \text{SS}(F_l)$ for $j \neq l$.

We also use the notation $\mathring{\text{SS}}(F) := \text{SS}(F) \cap \mathring{T}^*X = \text{SS}(F) \setminus T_X^*X$.

We denote by $\mathbf{D}(X)$ the (unbounded) derived category of sheaves of k -vector spaces on X . An object $F \in \mathbf{D}(X)$ is said to be locally bounded if for any relatively compact open subset U of X , one has $F|_U \in \mathbf{D}^b(U)$. We denote by $\mathbf{D}^{\text{lb}}(X)$ the full subcategory of $\mathbf{D}(X)$ consisting of locally bounded objects. A microsupport of an object of $\mathbf{D}^{\text{lb}}(X)$ can be defined in a totally same way as in Definition 2.1, since it is a local notion.

Example 2.2. (i) If F is a non-zero local system on a connected manifold X , then $\text{SS}(F) = T_X^*X$. Conversely, if $\text{SS}(F) \subset T_X^*X$ then the cohomology sheaves $H^i(F)$ are local systems for all $i \in \mathbb{Z}$.

(ii) If X is a smooth submanifold of a manifold Y , then $\text{SS}(k_X) = T_X^*Y \subset T^*Y$.

(iii) Let φ be a C^∞ -function and assume that $d\varphi(x)$ whenever $\varphi(x) = 0$. Let $U := \{x \in X \mid \varphi(x) > 0\}$ and $Z := \{x \in X \mid \varphi(x) \geq 0\}$. Then

$$\begin{aligned} \text{SS}(k_U) &= T_X^*X|_U \cup \{(x; \lambda d\varphi(x)) \mid \varphi(x) = 0, \lambda \leq 0\}, \\ \text{SS}(k_Z) &= T_X^*X|_Z \cup \{(x; \lambda d\varphi(x)) \mid \varphi(x) = 0, \lambda \geq 0\}. \end{aligned} \quad (2.4)$$

Next we recall the microlocal Morse theory and Morse inequality for sheaves. Let $\varphi: X \rightarrow \mathbb{R}$ be a C^∞ -function. We set

$$\Gamma_{d\varphi} = \text{Graph}(d\varphi) := \{(x; d\varphi(x)) \mid x \in X\} \subset T^*X. \quad (2.5)$$

The following proposition is called (a particular case of) the microlocal Morse lemma. See [KS90, Proposition 5.4.17 and Corollary 5.4.19] for more details. The classical theory corresponds to the case F is the constant sheaf k_X .

Proposition 2.3. *Let $F \in \mathbf{D}^b(X)$ and $\varphi: X \rightarrow \mathbb{R}$ be a C^∞ -function. Assume that $\text{Supp}(F) \cap \varphi^{-1}((-\infty, t])$ is compact for any $t \in \mathbb{R}$. Let $a < b$ in \mathbb{R} or $a \in \mathbb{R}, b = +\infty$ and assume that $d\varphi(x) \notin \text{SS}(F)$ for any $x \in \varphi^{-1}([a, b))$. Then the canonical morphism*

$$R\Gamma(\varphi^{-1}((-\infty, b)); F) \longrightarrow R\Gamma(\varphi^{-1}((-\infty, a)); F) \quad (2.6)$$

is an isomorphism.

The Morse inequality can also be extended to sheaves as follows. For a bounded complex W of k -vector spaces with finite-dimensional cohomology, we set

$$b_j(W) := \dim H^j(W), \quad b_l^*(W) := (-1)^l \sum_{j \leq l} (-1)^j b_j(W). \quad (2.7)$$

Let $F \in \mathbf{D}^b(X)$ and $\varphi: X \rightarrow \mathbb{R}$ a C^∞ -function. We consider the following assumptions

- (1) $\text{Supp}(F) \cap \varphi^{-1}((-\infty, t])$ is compact for any $t \in \mathbb{R}$,
- (2) the set $\Gamma_{d\varphi} \cap \text{SS}(F)$ is finite: $\Gamma_{d\varphi} \cap \text{SS}(F) = \{p_1, \dots, p_N\}$,
- (3) if one sets $x_i := \pi(p_i)$, $W_i := R\Gamma_{\{\varphi \geq \varphi(x_i)\}}(F)_{x_i}$ has finite-dimensional cohomology for all $i = 1, \dots, N$.

Theorem 2.4 ([KS90, Proposition 5.4.20]). *Assume that (1)–(3) are satisfied. Then*

- (i) $R\Gamma(X; F)$ has finite-dimensional cohomology,
- (ii) one has

$$b_l^*(R\Gamma(X; F)) \leq \sum_{i=1}^N b_l^*(W_i) \quad (2.8)$$

for any $l \in \mathbb{Z}$.

Note that (2.8) implies

$$b_l(R\Gamma(X; F)) \leq \sum_{i=1}^N b_l(W_i) \quad (2.9)$$

for any $l \in \mathbb{Z}$.

2.3 Functorial operations ([KS90, §5.4])

We consider bounds for microsupports of proper direct images, non-characteristic inverse images, and $R\mathcal{H}om$.

Definition 2.5. Let $f: X \rightarrow Y$ be a morphism of manifolds and $A \subset T^*Y$ a closed conic subset. The morphism f is said to be *non-characteristic* for A if

$$f_\pi^{-1}(A) \cap T_X^*Y \subset X \times_Y T_Y^*Y. \quad (2.10)$$

See (2.1) for the notations f_π and f_d . In particular, if f is a submersion, f is non-characteristic for any closed conic subset in T^*Y . Note that submersions are called smooth morphisms in [KS85, KS90]. One can show that if f is non-characteristic for A , then $f_d f_\pi^{-1}(A)$ is a conic closed subset in T^*X .

Theorem 2.6 ([KS90, Proposition 5.4.4 and Proposition 5.4.13]). *Let $f: X \rightarrow Y$ be a morphism of manifolds, $F \in \mathbf{D}^b(X)$, and $G \in \mathbf{D}^b(Y)$.*

- (i) *Assume that f is proper on $\text{Supp}(F)$. Then $\text{SS}(Rf_*F) \subset f_\pi f_d^{-1}(\text{SS}(F))$.*
- (ii) *Assume that f is non-characteristic for $\text{SS}(G)$. Then the canonical morphism $f^{-1}G \otimes \omega_{X/Y} \rightarrow f^!G$ is an isomorphism and $\text{SS}(f^{-1}G) \cup \text{SS}(f^!G) \subset f_d f_\pi^{-1}(\text{SS}(G))$.*

Proposition 2.7 ([KS90, Proposition 5.4.2]). *Let X_i ($i = 1, 2$) be manifolds and denote by q_i the projection $X_1 \times X_2 \rightarrow X_i$. Let $F_i \in \mathbf{D}^b(X_i)$. Then one has*

$$\text{SS}(R\mathcal{H}om(q_2^{-1}F_2, q_1^{-1}F_1)) \subset \text{SS}(F_1) \times \text{SS}(F_2)^a. \quad (2.11)$$

Using Proposition 2.7 and Theorem 2.6 (ii) for the diagonal embedding $\delta: X \rightarrow X \times X$, one can prove the following.

Proposition 2.8 ([KS90, Proposition 5.4.14 (ii)]). *Let $F, G \in \mathbf{D}^b(X)$ and assume that $\text{SS}(F) \cap \text{SS}(G) \subset T_X^*X$. Then*

$$\text{SS}(R\mathcal{H}om(F, G)) \subset \text{SS}(F)^a + \text{SS}(G), \quad (2.12)$$

where $+$ is the fiberwise sum.

2.4 Non-proper direct image ([Tam08] and [GS14])

We consider estimates of microsupports of non-proper direct images for special cases. Let V_1 and V_2 be vector spaces and $u: X \times V_1 \rightarrow X \times V_2$ a constant linear map, that is, the map of the form $\text{id}_X \times \tilde{u}$ with $\tilde{u}: V_1 \rightarrow V_2$ is a linear map of vector spaces. The map u induces the maps

$$\begin{array}{ccc} & T^*X \times V_1 \times V_2^* & \\ u_d \swarrow & & \searrow u_\pi \\ T^*X \times V_1 \times V_1^* & & T^*X \times V_2 \times V_2^* \\ v_\pi \searrow & & \swarrow v_d \\ & T^*X \times V_2 \times V_1^* & \end{array} \quad (2.13)$$

Note that for a subset A of $T^*(X \times V_1)$ we have $u_\pi(u_d^{-1}(A)) = v_d^{-1}(v_\pi(A))$.

Notation 2.9. Let u be as above and $A \subset T^*(X \times V_1)$ a closed subset. One sets

$$u_\#(A) := v_d^{-1}(\overline{v_\pi(A)}). \quad (2.14)$$

Proposition 2.10 ([Tam08, Lemma 3.3] and [GS14, Theorem 1.16]). *Let u be as above and $F \in \mathbf{D}^b(X \times V_1)$. Then*

$$\text{SS}(Ru_*F) \cup \text{SS}(Ru_!F) \subset u_\#(\text{SS}(F)). \quad (2.15)$$

2.5 Kernels ([KS90, §3.6])

Let X_i ($i = 1, 2, 3$) be manifolds. We write $X_{ij} := X_i \times X_j$ and $X_{123} := X_1 \times X_2 \times X_3$ for short. We use the same symbol q_i for the projections $X_{ij} \rightarrow X_i$ and $X_{123} \rightarrow X_i$. We also denote by q_{ij} the projection $X_{123} \rightarrow X_{ij}$. Similarly, we denote by p_{ij} the projection $T^*X_{123} \rightarrow T^*X_{ij}$. One denotes by p_{12^a} the composite of p_{12} and the antipodal map on T^*X_2 .

Let $A \subset T^*X_{12}$ and $B \subset T^*X_{23}$. We set

$$A \circ B := p_{13}(p_{12^a}^{-1}A \cap p_{23}^{-1}B) \subset T^*X_{13}. \quad (2.16)$$

We define the operations of composition of kernels as follows:

$$\begin{aligned} \circ_{X_2} : \mathbf{D}^b(X_{12}) \times \mathbf{D}^b(X_{23}) &\rightarrow \mathbf{D}^b(X_{13}) \\ (K_{12}, K_{23}) &\mapsto K_{12} \circ_{X_2} K_{23} := Rq_{13!}(q_{12}^{-1}K_{12} \otimes q_{23}^{-1}K_{23}). \end{aligned} \quad (2.17)$$

Let $\Lambda_{ij} := \text{SS}(K_{ij}) \subset T^*X_{ij}$ ($ij = 12, 23$) and assume that

- (i) q_{13} is proper on $q_{12}^{-1}\text{Supp}(K_{12}) \cap q_{23}^{-1}\text{Supp}(K_{23})$,
- (ii) $p_{12^a}^{-1}\Lambda_{12} \cap p_{23}^{-1}\Lambda_{23} \cap (T_{X_1}^*X_1 \times T^*X_2 \times T_{X_3}^*X_3) \subset T_{X_{123}}^*X_{123}$.

Then by Theorem 2.6 and estimates of microsupports of tensor products (see [KS90, Proposition 5.4.14]), we have

$$\text{SS}(K_{12} \circ_{X_2} K_{23}) \subset \Lambda_{12} \circ \Lambda_{23}. \quad (2.18)$$

If there is no risk of confusion, we simply write \circ instead of \circ_{X_2} .

2.6 Localization ([KS90, §6.1])

Let $A \subset T^*X$ be a subset and set $\Omega = T^*X \setminus A$. We denote by $\mathbf{D}_A^b(X)$ the subcategory of $\mathbf{D}^b(X)$ consisting of sheaves whose microsupports are contained in A . By the triangular inequality, the subcategory $\mathbf{D}_A^b(X)$ is a triangulated subcategory. We set

$$\mathbf{D}^b(X; \Omega) := \mathbf{D}^b(X) / \mathbf{D}_A^b(X), \quad (2.19)$$

the localization of $\mathbf{D}^b(X)$ by $\mathbf{D}_A^b(X)$. A morphism $u: F \rightarrow G$ in $\mathbf{D}^b(X)$ becomes an isomorphism in $\mathbf{D}^b(X; \Omega)$ if u is embedded in a distinguished triangle $F \rightarrow G \rightarrow H \xrightarrow{+1}$ with $\text{SS}(H) \cap \Omega = \emptyset$. For a closed subset B of Ω , $\mathbf{D}_B^b(X; \Omega)$ denotes the full triangulated subcategory of $\mathbf{D}^b(X; \Omega)$ consisting of F with $\text{SS}(F) \cap \Omega \subset B$. In the case $\Omega = \{p\}$ for $p \in T^*X$, we simply write $\mathbf{D}^b(X; p)$ instead of $\mathbf{D}^b(X; \{p\})$.

Note that our notation is the same as in [KS90] and slightly differs from that of [Gui12, Gui16].

By using the following “refined microlocal cut-off lemma”, we can construct an isomorphic object in the localized category with an estimate of the microsupport. Recall that a cone is said to be proper if it contains no lines.

Proposition 2.11 ([KS90, Proposition 6.1.4]). *Let $x_0 \in X$, K a proper closed cone of $T_{x_0}^*X$, and $U \subset K$ an open cone. Let $F \in \mathbf{D}^b(X)$ and W a conic neighborhood of $K \cap \text{SS}(F) \setminus \{0\}$. Then there exist F' and a morphism $u: F' \rightarrow F$ satisfying the following conditions:*

- (i) u is an isomorphism in $\mathbf{D}^b(X; U)$,
- (ii) $\pi^{-1}(x_0) \cap \mathrm{SS}(F') \subset W \cup \{0\}$.

Next we recall the notion of microlocal inverse images. Let $f: X \rightarrow Y$ be a morphism of manifolds, $p \in X \times_Y T^*Y$. We set $p_X := f_d(p) \in T^*X$, $p_Y := f_\pi(p) \in T^*Y$.

Definition 2.12 ([KS90, Definition 6.1.7 (ii)]). Let $G \in \mathbf{D}^b(Y; p_Y)$. The microlocal inverse image of G at p is defined by the pro-object of $\mathbf{D}^b(X; p_X)$

$$f_{\mu,p}^{-1}G := \varprojlim_{G' \rightarrow G} f^{-1}G', \quad (2.20)$$

where $G' \rightarrow G$ ranges over morphisms in $\mathbf{D}^b(Y)$ which are isomorphisms in $\mathbf{D}^b(Y; p_Y)$.

In [KS90], the authors simply wrote f_μ^{-1} , but in this paper we use the notation $f_{\mu,p}^{-1}$ in order to emphasize the dependence on the point p .

In this paper, we only consider the cases where the microlocal inverse images belong to $\mathbf{D}^b(X; p_X)$. The next proposition is a slight generalization of [KS90, Proposition 6.1.9] (though it is almost written in loc. cit.).

Proposition 2.13. (i) *Let $G \in \mathbf{D}^b(Y; p_Y)$. If $f_d^{-1}(p_X) \cap f_\pi^{-1}(\mathrm{SS}(G)) \subset \{p\}$ in a neighborhood of p , then $f_{\mu,p}^{-1}G$ belongs to $\mathbf{D}^b(X; p_X)$. Moreover for any neighborhood W of p , one has*

$$\mathrm{SS}(f_{\mu,p}^{-1}G) \subset f_d(f_\pi^{-1}(\mathrm{SS}(G)) \cap W) \quad \text{in a neighborhood of } p_X. \quad (2.21)$$

(ii) *Let $G \in \mathbf{D}^b(Y)$ and $p_X \in \mathring{T}^*X$. Assume the following conditions:*

- (a) f is non-characteristic for $\mathrm{SS}(G)$,
- (b) *there exist finitely many points $p_1, \dots, p_{N'} \in X \times_Y \mathring{T}^*Y$ such that*

$$f_d^{-1}(p_X) \cap f_\pi^{-1}(\mathrm{SS}(G)) \subset \{p_1, \dots, p_{N'}\}. \quad (2.22)$$

Then one has an isomorphism

$$f^{-1}G \simeq \bigoplus_{j=1}^{N'} f_{\mu,p_j}^{-1}G \quad \text{in } \mathbf{D}^b(X; p_X). \quad (2.23)$$

Proof. (i) is the same as [KS90, Proposition 6.1.9 (i)].

(ii) We shall use a similar argument to the proof of [KS90, Proposition 6.1.9]. Set $p_{Y,j} := f_\pi(p_j) \in \mathring{T}^*Y$, $x_0 := \pi_X(p_X) \in X$, $y_0 := f(x_0) (= \pi_Y(p_{Y,j})) \in Y$, $V := \mathrm{Ker}(T_{y_0}^*Y \rightarrow T_{x_0}^*X)$. Take proper closed convex cones K_j and open convex cones U_j such that $K_i \cap K_j \setminus \{0\} = \emptyset$ ($i \neq j$), $p_{Y,j} \in U_j \subset K_j$, $K_j \cap V \subset \{0\}$, and $f_\pi f_d^{-1}(p_X) \cap \mathrm{SS}(G) \cap K_j \subset \{p_{Y,j}\}$.

By the refined microlocal cut-off lemma (Proposition 2.11), there exist morphisms $u_j: G'_j \rightarrow G$ ($j = 1, \dots, N'$) such that u_j is an isomorphism in $\mathbf{D}^b(Y; p_{Y,j})$ and G'_j satisfies the conditions

- (a) f is non-characteristic for $\mathrm{SS}(G'_j)$,
- (b) $f_d^{-1}(p_X) \cap f_\pi^{-1}(\mathrm{SS}(G'_j)) \subset \{p_j\}$

for any $j = 1, \dots, N'$. Embedding the morphism $\bigoplus_j G'_j \rightarrow G$ into a distinguished triangle $\bigoplus_j G'_j \rightarrow G \rightarrow G_0 \xrightarrow{+1}$, we find that f is non-characteristic for $\mathrm{SS}(G_0)$ by the triangular inequality. Moreover we get $f_d^{-1}(p_X) \cap f_\pi^{-1}(\mathrm{SS}(G_0)) = \emptyset$. Hence by Theorem 2.6 (ii), $p_X \notin \mathrm{SS}(f^{-1}F_0)$, which shows $\bigoplus_j f^{-1}G'_j \rightarrow f^{-1}G$ is an isomorphism in $\mathbf{D}^b(X; p_X)$. It remains to apply [KS90, Proposition 6.1.9 (ii)] to each G'_j . \square

2.7 Simple sheaves ([KS85, Chapter 7] and [KS90, §7.5])

Let $\Lambda \subset \overset{\circ}{T}^*X$ be a locally closed Lagrangian submanifold and $p \in \Lambda$. Pure and simple sheaves along Λ at p are defined in [KS90, Definition 7.5.4]. In this subsection, we briefly recall them.

Definition 2.14. Let $p \in \Lambda$ and $\varphi: X \rightarrow \mathbb{R}$ be a C^∞ -function on X . The function φ is said to be transversal to Λ at p if $\varphi(\pi(p)) = 0$ and the manifolds Λ and $\Gamma_{d\varphi}$ intersect transversally at p .

For $p \in \Gamma_{d\varphi} \cap \Lambda$, we define the following Lagrangian subspaces in $T_p T^*X$:

$$\lambda_0(p) := T_p T_{\pi(p)}^* X, \quad \lambda_\Lambda(p) := T_p \Lambda, \quad \lambda_\varphi(p) := T_p \Gamma_{d\varphi}. \quad (2.24)$$

Recall the definition of the inertia index (see [KS85, §7.1] and [KS90, §A.3]). Let (E, σ) be a symplectic vector space and let $\lambda_1, \lambda_2, \lambda_3$ three Lagrangian subspaces of E . We define a quadratic form q on $\lambda_1 \oplus \lambda_2 \oplus \lambda_3$ by $q(v_1, v_2, v_3) = \sigma(v_1, v_2) + \sigma(v_2, v_3) + \sigma(v_3, v_1)$. Then $\tau_E(\lambda_0, \lambda_1, \lambda_3)$ is defined as the signature of q . One sets

$$\tau_\varphi = \tau_{p, \varphi} := \tau_{T_p T^*X}(\lambda_0(p), \lambda_\Lambda(p), \lambda_\varphi(p)). \quad (2.25)$$

Proposition 2.15 ([KS90, Proposition 7.5.3]). *Let $\varphi_1, \varphi_2: X \rightarrow \mathbb{R}$ be C^∞ -functions transversal to Λ at p . Let $F \in \mathbf{D}^b(X)$ and assume that $\text{SS}(F) \subset \Lambda$ in a neighborhood of p . Then*

$$R\Gamma_{\{\varphi_1 \geq 0\}}(F)_{\pi(p)} \simeq R\Gamma_{\{\varphi_2 \geq 0\}}(F)_{\pi(p)} \left[\frac{1}{2}(\tau_{\varphi_2} - \tau_{\varphi_1}) \right]. \quad (2.26)$$

Definition 2.16 ([KS90, Definition 7.5.4]). In the situation of Proposition 2.15, F has microlocal type $L \in \mathbf{D}^b(\text{Mod}(k))$ with shift $d \in \frac{1}{2}\mathbb{Z}$ at p if

$$R\Gamma_{\{\varphi \geq 0\}}(F)_{\pi(p)} \simeq L \left[d - \frac{1}{2}d_X - \frac{1}{2}\tau_\varphi \right] \quad (2.27)$$

for some (hence for any) C^∞ -function φ transversal to Λ at p . If $L \in \text{Mod}(k)$, F is said to be *pure* along Λ at p . If moreover $L \simeq k$, F is said to be *simple* along Λ at p .

If F is pure (resp. simple) at all points of Λ , one says that F is pure (resp. simple) along Λ .

When Λ is a conormal bundle to a submanifold $Y \subset X$ in a neighborhood of p , that is, $\pi|_\Lambda: \Lambda \rightarrow X$ has constant rank, then $F \in \mathbf{D}^b(X)$ is simple along Λ at p if $F \simeq k_Y[d]$ in $\mathbf{D}^b(X; p)$ for some $d \in \mathbb{Z}$.

Proposition 2.17 ([KS90, Proposition 7.5.10 (ii)]). *Let X_i ($i = 1, 2$) be manifolds, $\Lambda_i \subset T^*X_i$ a Lagrangian submanifold, and $p_i \in \Lambda_i$. Denote by q_i the projection $X_1 \times X_2 \rightarrow X_i$. Let $F_i \in \mathbf{D}^b(X_i)$ and assume that F_i has microlocal type L_i with shift d_i at p_i along Λ_i . Then $R\mathcal{H}om(q_2^{-1}F_2, q_1^{-1}F_1)$ has microlocal type $R\mathcal{H}om(L_2, L_1)$ with shift $d_1 - d_2$ at (p_1, p_2^a) along $\Lambda_1 \times \Lambda_2^a$.*

The microlocal inverse images preserve purity and simplicity under some assumptions.

Proposition 2.18 ([KS90, Corollary 7.5.13]). *Let $f: X \rightarrow Y$ be a morphism of manifolds, $p \in X \times_Y T^*Y$ and set $p_X := f_d(p) \in T^*X$, $p_Y := f_\pi(p) \in T^*Y$. Let Λ_Y be a Lagrangian submanifold of T^*Y such that f_π is transversal to Λ_Y at p_Y . Let $G \in \mathbf{D}^b(Y)$ and assume that $\text{SS}(G) \subset \Lambda_Y$ in a neighborhood of p_Y and F has microlocal type L with shift d along Λ_Y at p_Y .*

- (i) For a sufficiently small open neighborhood W of p , $f_d(f_\pi^{-1}(\Lambda_Y) \cap W)$ is a Lagrangian submanifold of T^*X , isomorphic to $f_\pi^{-1}(\Lambda_Y) \cap W$ by f_d .
- (ii) $f_{\mu,p}^{-1}G \in \mathbf{D}^b(X; p_X)$ has microlocal type L with shift d at p_X .

Next we consider the method to calculate stalks of local cohomology sheaves of direct images.

Proposition 2.19 (cf. [KS85, Theorem 7.3.1]). *Let $F \in \mathbf{D}^b(X)$ and $p_Y \in T^*Y$. Set $y_0 := \pi_Y(p_Y)$. Assume that*

- (1) f is proper on $\text{Supp}(F)$,
- (2) there exist finitely many points $p_1, \dots, p_N \in X \times_Y T^*Y$ such that

$$f_\pi^{-1}(p_Y) \cap f_d^{-1}(\text{SS}(F)) \subset \{p_1, \dots, p_N\}, \quad (2.28)$$

- (3) if one sets $p_{X,i} := f_d(p_i) \in T^*X$, for any $i = 1, \dots, N$, there exist smooth Lagrangian submanifolds $\Lambda_{X,i,j}$ containing $p_{X,i}$ and $F_{i,j} \in \mathbf{D}^b(X)$ ($j = 1, \dots, N'_i$) such that $F_{i,j}$ has microlocal type $L_{i,j}$ with shift $d_{i,j}$ along $\Lambda_{X,i,j}$ at $p_{X,i}$ and

$$F \simeq \bigoplus_{j=1}^{N'_i} F_{i,j} \quad \text{in } \mathbf{D}^b(X; p_{X,i}), \quad (2.29)$$

- (4) for any $i = 1, \dots, N$ and any $j = 1, \dots, N'_i$, f_d is transversal to $\Lambda_{X,i,j}$ at $p_{X,i}$.

Then one has the following:

- (i) For any $i = 1, \dots, N$, there exists a sufficiently small open neighborhood W_i of p_i such that $\Lambda_{Y,i,j} := f_\pi(f_d^{-1}(\Lambda_{X,i,j}) \cap W_i)$ is a Lagrangian submanifold of T^*Y for any $j = 1, \dots, N'_i$.
- (ii) Let φ be a C^∞ -function on Y transversal to $\Lambda_{Y,i,j}$ at $p_{Y,i}$ for all i and j . Then one has

$$R\Gamma_{\{\varphi \geq 0\}}(Rf_*F)_{y_0} \simeq \bigoplus_{i=1}^N \bigoplus_{j=1}^{N'_i} L_{i,j}[d'_{i,j}], \quad (2.30)$$

where the shifts $d'_{i,j} \in \mathbb{Z}$ are defined as follows:

$$d'_{i,j} := d_{i,j} - \frac{1}{2}d_X - \frac{1}{2}\tau_{p_Y, \varphi} - \frac{1}{2}\tau(\lambda_0(p_{X,i}), \lambda_{\Lambda_{X,i,j}}(p_{X,i}), f_d(p_i)f_\pi(p_i)^{-1}\lambda_0(p_Y)), \quad (2.31)$$

where $f_d(p_i)f_\pi(p_i)^{-1}\lambda_0(p_Y)$ means $df_d(p_i)df_\pi(p_i)^{-1}\lambda_0(p_Y)$.

Proof. (i) is the same as [KS90, Corollary 7.5.12 (i)].

(ii) The proof is similar to that of [KS85, Theorem 7.3.1]. Set $x_i := \pi_X(p_{X,i})$. By the assumption (4) and the transversality of φ , $\tilde{\varphi} := \varphi \circ f$ is transversal to $\Lambda_{X,i,j}$ at $p_{X,i}$. Hence by (3) we have

$$R\Gamma_{\{\tilde{\varphi} \geq 0\}}(F)_{x_i} \simeq \bigoplus_{j=1}^{N'_i} L_{i,j}[d_{i,j}] \quad (2.32)$$

for some shift $d_{i,j} \in \mathbb{Z}$. The assumption (2) implies

$$R\Gamma_{\{\tilde{\varphi} \geq 0\}}(F)_x \simeq 0 \quad \text{for } x \in f^{-1}(y_0) \setminus \{x_1, \dots, x_N\}. \quad (2.33)$$

Combining this with the assumption (1), we obtain

$$R\Gamma_{\{\varphi \geq 0\}}(Rf_*F)_{y_0} \simeq R\Gamma(f^{-1}(y_0); R\Gamma_{\{\tilde{\varphi} \geq 0\}}(F)) \simeq \bigoplus_{i=1}^N R\Gamma_{\{\tilde{\varphi} \geq 0\}}(F)_{x_i}. \quad (2.34)$$

For the shifts, we refer to [KS85, Theorem 7.3.1] and omit the details. \square

3 Sheaf quantization and Tamarkin's non-displaceability theorem

In what follows, until the end of the paper, let M be a non-empty compact connected manifold without boundary.

In this section, we review Tamarkin's approach to non-displaceability problems in symplectic geometry based on the microlocal sheaf theory. We also review sheaf quantization of Hamiltonian isotopies and compact exact Lagrangian submanifolds in a cotangent bundle.

3.1 Sheaf quantization of Hamiltonian isotopies ([GKS12])

Guillermou-Kashiwara-Schapira [GKS12] constructed a sheaf quantization of a Hamiltonian isotopy. Since microsupports of sheaves are conic subset of a cotangent bundle, the microlocal sheaf theory is related to the exact (homogeneous) symplectic structure rather than the symplectic structure of T^*M . In order to treat non-homogeneous Hamiltonian isotopies and non-conic Lagrangian submanifolds, an important trick is to add a variable and “conify” them.

We set $T_{\tau > 0}^*(M \times \mathbb{R}) := \{\tau > 0\} \subset T^*(M \times \mathbb{R})$ and define the map

$$\begin{array}{ccc} \rho: T_{\tau > 0}^*(M \times \mathbb{R}) & \longrightarrow & T^*M \\ \downarrow \Psi & & \downarrow \Psi \\ (x, t; \xi, \tau) & \longmapsto & (x; \xi/\tau). \end{array} \quad (3.1)$$

Let I be an open interval in \mathbb{R} containing 0 and $\phi = (\phi_s)_{s \in I}: T^*M \times I \rightarrow T^*M$ a Hamiltonian isotopy with compact support. Note that ϕ is the identity for $s = 0$: $\phi_0 = \text{id}_{T^*M}$. Then one can construct a homogeneous Hamiltonian isotopy $\hat{\phi}: \mathring{T}^*(M \times \mathbb{R}) \times I \rightarrow \mathring{T}^*(M \times \mathbb{R})$ such that the following diagram commutes:

$$\begin{array}{ccc} T_{\tau > 0}^*(M \times \mathbb{R}) \times I & \xrightarrow{\hat{\phi}} & T_{\tau > 0}^*(M \times \mathbb{R}) \\ \rho \times \text{id} \downarrow & & \downarrow \rho \\ T^*M \times I & \xrightarrow{\phi} & T^*M. \end{array} \quad (3.2)$$

Here $\hat{\phi}$ is called a homogeneous Hamiltonian isotopy if it is a Hamiltonian isotopy whose Hamiltonian function \hat{H} is homogeneous of degree 1: $\hat{H}_s(x, t; c\xi, c\tau) = c \cdot \hat{H}_s(x, t; \xi, \tau)$ for any $c > 0$. See [GKS12, Subsection A.3] for more details. For simplicity, we set $N := M \times \mathbb{R}$ and consider a homogeneous Hamiltonian isotopy $\hat{\phi} = (\hat{\phi}_s)_s: \mathring{T}^*N \times I \rightarrow \mathring{T}^*N$ and the

associated homogeneous Hamiltonian $\widehat{H}: \dot{T}^*N \times I \rightarrow \mathbb{R}$. We define a conic Lagrangian submanifold $\Lambda_{\widehat{\phi}} \subset \dot{T}^*N \times \dot{T}^*N \times T^*I$ by

$$\Lambda_{\widehat{\phi}} := \left\{ \left(\widehat{\phi}_s(y; \eta), (y; -\eta), (s; -\widehat{H}_s \circ \widehat{\phi}_s(y; \eta)) \right) \mid (y; \eta) \in \dot{T}^*N, s \in I \right\}. \quad (3.3)$$

Note that

$$\Lambda_{\widehat{\phi}} \circ T_s^*I = \widetilde{\Gamma}_{\widehat{\phi}_s} := \left\{ \left(\widehat{\phi}_s(y; \eta), (y; -\eta) \right) \mid (y; \eta) \in \dot{T}^*N \right\} \subset \dot{T}^*N \times \dot{T}^*N \quad (3.4)$$

for any $s \in I$ (see (2.16) for the definition of $A \circ B$). The set $\widetilde{\Gamma}_{\widehat{\phi}_s}$ is called the twisted graph of $\widehat{\phi}_s$.

Theorem 3.1 ([GKS12, Theorem 4.3]). *For a homogeneous Hamiltonian isotopy $\widehat{\phi}: \dot{T}^*N \times I \rightarrow \dot{T}^*N$, there exists a unique object $K \in \mathbf{D}^{\text{lb}}(N \times N \times \mathbb{R})$ satisfying the following conditions:*

- (i) $\text{SS}(K) \subset \Lambda_{\widehat{\phi}} \cup T_{N \times N \times I}^*(N \times N \times I)$,
- (ii) $K|_{s=0} \simeq k_{\Delta_N}$, where Δ_N is the diagonal of $N \times N$.

Moreover K is simple along $\Lambda_{\widehat{\phi}}$ and both projections $\text{Supp}(K) \rightarrow N \times I$ are proper.

The object K is called the *quantization* of $\widehat{\phi}$. Note also that $\text{SS}(K|_{s=s_0}) \subset \widetilde{\Gamma}_{\widehat{\phi}_{s_0}}$.

3.2 Tamarkin's non-displaceability theorem ([Tam08], after Guillermou-Schapira [GS14])

Tamarkin [Tam08] (see also Guillermou-Schapira [GS14]) considered the following categories, from which he deduced non-displaceability of Lagrangian submanifolds.

Denote by $(x; \xi)$ a local homogeneous coordinate system on T^*M and by $(t; \tau)$ the homogeneous coordinate system on $T^*\mathbb{R}$. Define the maps

$$\begin{aligned} q_1, q_2, s: M \times \mathbb{R} \times \mathbb{R} &\longrightarrow M \times \mathbb{R}, \\ q_1(x, t_1, t_2) &= (x, t_1), \quad q_2(x, t_1, t_2) = (x, t_2), \quad s(x, t_1, t_2) = (x, t_1 + t_2). \end{aligned} \quad (3.5)$$

We also use the notation

$$i: M \times \mathbb{R} \rightarrow M \times \mathbb{R}, \quad (x, t) \longmapsto (x, -t). \quad (3.6)$$

Definition 3.2. For $F, G \in \mathbf{D}^{\text{b}}(M \times \mathbb{R})$, one sets

$$F \star G := R s_! (q_1^{-1} F \otimes q_2^{-1} G), \quad (3.7)$$

$$\mathcal{H}om^*(F, G) := R q_{1*} R \mathcal{H}om(q_2^{-1} F, s^! G) \quad (3.8)$$

$$\simeq R s_* R \mathcal{H}om(q_2^{-1} i^{-1} F, q_1^! G). \quad (3.9)$$

Note that the functor \star is a left adjoint to $\mathcal{H}om^*$. The functor

$$k_{M \times [0, +\infty)} \star (\cdot): \mathbf{D}^{\text{b}}(M \times \mathbb{R}) \longrightarrow \mathbf{D}^{\text{b}}(M \times \mathbb{R}) \quad (3.10)$$

defines a projector on the left orthogonal ${}^{\perp} \mathbf{D}_{\{\tau \leq 0\}}^{\text{b}}(M \times \mathbb{R})$. By using this projector, Tamarkin proved that the localized category $\mathbf{D}^{\text{b}}(M \times \mathbb{R}; \{\tau > 0\})$ is equivalent to the left orthogonal ${}^{\perp} \mathbf{D}_{\{\tau \leq 0\}}^{\text{b}}(M \times \mathbb{R})$:

$$\mathbf{D}^{\text{b}}(M \times \mathbb{R}; \{\tau > 0\}) = \mathbf{D}^{\text{b}}(M \times \mathbb{R}) / \mathbf{D}_{\{\tau \leq 0\}}^{\text{b}}(M \times \mathbb{R}) \xrightarrow{\sim} {}^{\perp} \mathbf{D}_{\{\tau \leq 0\}}^{\text{b}}(M \times \mathbb{R}). \quad (3.11)$$

We denote by $\mathcal{D}(M)$ the quotient category $\mathbf{D}^{\text{b}}(M \times \mathbb{R}; \{\tau > 0\})$ for short. For an object of $\mathcal{D}(M)$, we always take the canonical representative in ${}^{\perp} \mathbf{D}_{\{\tau \leq 0\}}^{\text{b}}(M \times \mathbb{R})$ via the projector.

Proposition 3.3 ([GS14, Lemma 3.18]). *For $F, G \in \mathcal{D}(M)$, one has an isomorphism*

$$\mathrm{Hom}_{\mathcal{D}(M)}(F, G) \simeq H_{M \times [0, +\infty)}^0(M \times \mathbb{R}; \mathcal{H}om^*(F, G)). \quad (3.12)$$

Tamarkin [Tam08] proved the non-displaceability theorem by using the category $\mathcal{D}(M)$ and the notion of torsion objects. Moreover Guillermou-Schapira [GS14] proved that torsion objects form a triangulated subcategory and introduced the quotient category $\mathcal{T}(M)$, which we will explain below.

For a compact subset A of T^*M , we define a full subcategory $\mathcal{D}_A(M)$ of $\mathcal{D}(M)$ by

$$\mathcal{D}_A(M) := \mathbf{D}_{\rho^{-1}(A)}^b(M \times \mathbb{R}; \{\tau > 0\}). \quad (3.13)$$

The following separation theorem is due to Tamarkin [Tam08].

Theorem 3.4 ([Tam08, Theorem 3.2] and [GS14, Theorem 3.28]). *Let A and B be compact subsets of T^*M and assume that $A \cap B = \emptyset$. Then for $F \in \mathcal{D}_A(M)$ and $G \in \mathcal{D}_B(M)$, one has $\mathrm{Hom}_{\mathcal{D}(M)}(F, G) \simeq 0$.*

Proof. We give the outline of the proof of the theorem due to Guillermou-Schapira [GS14] (a new proof of Tamarkin's result). Denote by $t: M \times \mathbb{R} \rightarrow \mathbb{R}$ the function $(x, t) \mapsto t$. Recall the notation $\Gamma_{dt} = \mathrm{Graph}(dt) = \{(x, t; 0, 1)\}$. Then one can show that

$$\Gamma_{dt} \cap \mathrm{SS}(R\Gamma_{M \times [0, +\infty)} \mathcal{H}om^*(F, G)) = \emptyset. \quad (3.14)$$

Hence by Proposition 3.3 and the microlocal Morse lemma (Proposition 2.3), we have the conclusion. \square

For non-displaceability problems, we need a Hamiltonian invariance property. One can get such invariance by taking the quotient by torsion objects. Define the translation map

$$T_c: M \times \mathbb{R} \rightarrow M \times \mathbb{R}; (x, t) \mapsto (x, t + c). \quad (3.15)$$

For $F \in {}^\perp \mathbf{D}_{\{\tau \leq 0\}}^b(M \times \mathbb{R})$, there exists a canonical morphism

$$\tau_{0,c}(F): F \longrightarrow T_{c*}F. \quad (3.16)$$

Definition 3.5 (Tamarkin). An object $F \in {}^\perp \mathbf{D}_{\{\tau \leq 0\}}^b(M \times \mathbb{R})$ is said to be a *torsion object* if $\tau_{0,c}(F) = 0$ for some $c \geq 0$. Denote by $\mathcal{N}_{\mathrm{tor}}$ the subcategory of torsion objects in ${}^\perp \mathbf{D}_{\{\tau \leq 0\}}^b(M \times \mathbb{R}) \simeq \mathcal{D}(M)$.

Let $F \in {}^\perp \mathbf{D}_{\{\tau \leq 0\}}^b(M \times \mathbb{R})$ and assume that $\mathrm{Supp}(F) \subset M \times [a, b]$ for some compact interval $[a, b]$ of \mathbb{R} . Then F is a torsion object.

Proposition 3.6 ([GS14, Theorem 5.4]). *The subcategory $\mathcal{N}_{\mathrm{tor}}$ is a full triangulated subcategory.*

Definition 3.7 ([GS14, Definition 5.6]). The triangulated category $\mathcal{T}(M)$ is the quotient category of $\mathcal{D}(M)$ by $\mathcal{N}_{\mathrm{tor}}$: $\mathcal{T}(M) := \mathcal{D}(M)/\mathcal{N}_{\mathrm{tor}}$.

Hom spaces in $\mathcal{T}(M)$ can be described as inductive limits of those in $\mathcal{D}(M)$.

Proposition 3.8 ([GS14, Proposition 5.7]). *For $F, G \in \mathcal{D}(M)$, one has an isomorphism*

$$\lim_{c \rightarrow +\infty} \mathrm{Hom}_{\mathcal{D}(M)}(F, T_{c*}G) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{T}(M)}(F, G). \quad (3.17)$$

Recall that for a symplectic manifold X , a diffeomorphism $\psi: X \rightarrow X$ is said to be a Hamiltonian diffeomorphism if there exists a Hamiltonian isotopy with compact support $\phi = (\phi_s)_s: X \times [0, 1] \rightarrow X$ such that $\phi_1 = \psi$ and $\phi_0 = \text{id}_X$. Let $\psi: T^*M \rightarrow T^*M$ be a Hamiltonian diffeomorphism and $\phi = (\phi_s)_s: T^*M \times I \rightarrow T^*M$ a Hamiltonian isotopy satisfying the above conditions, where I is an open interval containing $[0, 1]$. Let $\hat{\phi}: \dot{T}^*(M \times \mathbb{R}) \times I \rightarrow \dot{T}^*(M \times \mathbb{R})$ be the associated homogeneous Hamiltonian isotopy and $K \in \mathbf{D}^{\text{lb}}(M \times \mathbb{R} \times M \times \mathbb{R} \times I)$ the quantization of $\hat{\phi}$. Then the composition with $K_1 := K|_{s=1} \in \mathbf{D}^{\text{b}}(M \times \mathbb{R} \times M \times \mathbb{R})^2$ defines a functor

$$\Psi = K_1 \circ (*): \mathbf{D}^{\text{b}}(M \times \mathbb{R}) \longrightarrow \mathbf{D}^{\text{b}}(M \times \mathbb{R}), \quad (3.18)$$

which induces a functor $\Psi: \mathcal{D}(M) \rightarrow \mathcal{D}(M)$ (see [GS14, Proposition 3.29]). Moreover for a compact subset A of T^*M and $F \in \mathcal{D}_A(M)$, the commutative diagram (3.2) implies that

$$\text{SS}(K_1 \circ F) \cap \{\tau > 0\} \subset \tilde{\Gamma}_{\hat{\phi}_1} \circ \rho^{-1}(A) = \hat{\phi}_1(\rho^{-1}(A)) \subset \rho^{-1}(\psi(A)). \quad (3.19)$$

Hence the functor also induces $\Psi: \mathcal{D}_A(M) \rightarrow \mathcal{D}_{\psi(A)}(M)$. The following is a Hamiltonian invariance theorem due to Tamarkin [Tam08].

Theorem 3.9 ([Tam08, Theorem 3.9] and [GS14, Theorem 6.1]). *Let $F \in \mathcal{D}_A(M)$ and $\Psi: \mathcal{D}_A(M) \rightarrow \mathcal{D}_{\psi(A)}(M)$ be as above. Then one has an isomorphism*

$$F \simeq \Psi(F) \quad \text{in } \mathcal{T}(M). \quad (3.20)$$

Combining Theorem 3.9 with Theorem 3.4 and Proposition 3.8, we can deduce the non-displaceability theorem. Two compact subsets A and B of T^*M are said to be *mutually non-displaceable* if for any Hamiltonian diffeomorphism $\psi: T^*M \rightarrow T^*M$, $A \cap \psi(B) \neq \emptyset$.

Theorem 3.10 ([Tam08, Theorem 3.1] and [GS14, Corollary 6.3]). *Let A and B be compact subsets of T^*M . Assume that there exist $F \in \mathcal{D}_A(M)$ and $G \in \mathcal{D}_B(M)$ such that $\text{Hom}_{\mathcal{T}(M)}(F, G) \neq 0$. Then A and B are mutually non-displaceable.*

3.3 Guillermou's sheaf quantization of compact exact Lagrangian submanifolds ([Gui12, Gui16])

Guillermou [Gui12, Gui16] proved the existence of a sheaf quantization of a compact exact Lagrangian submanifold of T^*M . Let L be a compact connected exact Lagrangian submanifold of T^*M and choose a primitive of the Liouville form $f: L \rightarrow \mathbb{R}$ satisfying

$$\alpha|_L = df. \quad (3.21)$$

With the pair (L, f) , we associate the conification $\hat{L}_f \subset T_{\tau>0}^*(M \times \mathbb{R})$ of L with respect to f by

$$\hat{L}_f := \{(x, t; \tau\xi, \tau) \mid \tau > 0, (x; \xi) \in L, t = -f(x; \xi)\}. \quad (3.22)$$

If there is no risk of confusion, we simply write \hat{L} instead of \hat{L}_f .

²Although $\hat{\phi}$ does not satisfy [GKS12, (3.3)] in general, $K|_{M \times \mathbb{R} \times M \times \mathbb{R} \times J}$ is bounded for any relatively compact subinterval J of I . The author learned the detailed proof from S. Guillermou. One can prove it using the properness of $\text{Supp}(K) \rightarrow M \times \mathbb{R} \times I$ and the fact that $K \simeq \sigma^{-1}K'$, where $K' \in \mathbf{D}^{\text{lb}}(M \times M \times \mathbb{R} \times I)$ and $\sigma: M \times \mathbb{R} \times M \times \mathbb{R} \times I \rightarrow M \times M \times \mathbb{R} \times I$, $(x, t, x', t', s) \mapsto (x, x', t - t', s)$.

Consider the category $\mathbf{D}_{\widehat{L} \cup T_{M \times \mathbb{R}}^*(M \times \mathbb{R})}^b(M \times \mathbb{R})$ consisting of sheaves microsupported in $\widehat{L} \cup T_{M \times \mathbb{R}}^*(M \times \mathbb{R})$. By the compactness of L , there is $A > 0$ such that

$$\widehat{L} \subset T_{\tau > 0}^*(M \times (-A, A)). \quad (3.23)$$

Hence for any $F \in \mathbf{D}_{\widehat{L} \cup T_{M \times \mathbb{R}}^*(M \times \mathbb{R})}^b(M \times \mathbb{R})$, the restrictions $F|_{M \times (-\infty, -A)}$ and $F|_{M \times (A, +\infty)}$ are locally constant.

Definition 3.11 ([Gui12, Definition 20.1] and [Gui16, Definition 13.1]). For an object $F \in \mathbf{D}_{\widehat{L} \cup T_{M \times \mathbb{R}}^*(M \times \mathbb{R})}^b(M \times \mathbb{R})$, one defines $F_-, F_+ \in \mathbf{D}^b(M)$ by

$$F_- := F|_{M \times \{-t\}}, \quad F_+ := F|_{M \times \{t\}} \quad (3.24)$$

for any $t > A$ (independent of t). We denote by $\mathbf{D}_{\widehat{L} \cup T_{M \times \mathbb{R}}^*(M \times \mathbb{R}), +}^b(M \times \mathbb{R})$ the full subcategory of $\mathbf{D}_{\widehat{L} \cup T_{M \times \mathbb{R}}^*(M \times \mathbb{R})}^b(M \times \mathbb{R})$ consisting of F such that $F_- \simeq 0$.

Guillermou [Gui12, Gui16] proved the following existence and uniqueness of a canonical sheaf quantization of a compact exact Lagrangian submanifold in a cotangent bundle.

Theorem 3.12 ([Gui12, Theorem 26.1] and [Gui16, Theorem 18.1]). *Let L and $\widehat{L} = \widehat{L}_f$ be as above.*

- (i) *There exists an object $F_{\widehat{L}} \in \mathbf{D}_{\widehat{L} \cup T_{M \times \mathbb{R}}^*(M \times \mathbb{R}), +}^b(M \times \mathbb{R})$ satisfying $(F_{\widehat{L}})_+ \simeq k_M$.*
- (ii) *Moreover $F_{\widehat{L}}$ in (i) is unique up to a unique isomorphism and simple along \widehat{L} .*

We call $F_{\widehat{L}}$ the *canonical quantization* of \widehat{L} .

4 Intersection of compact exact Lagrangians in cotangent bundles and sheaf quantization

In this section, we study intersection of two compact exact Lagrangians in cotangent bundles using the results presented in the previous sections.

Let L_i ($i = 1, 2$) be compact connected exact Lagrangian submanifolds of T^*M and choose primitives $f_i: L_i \rightarrow \mathbb{R}$ satisfying (3.21). For the pair (L_i, f_i) we define the conification \widehat{L}_i of L_i with respect to f_i as in (3.22). By Theorem 3.12, there exist the canonical quantizations $F_i \in \mathbf{D}_{\widehat{L}_i \cup T_{M \times \mathbb{R}}^*(M \times \mathbb{R})}^b(M \times \mathbb{R})$ satisfying the conditions (i) and (ii).

4.1 Hom spaces in $\mathcal{T}(M)$ associated with two compact exact Lagrangians

First we prove $\mathrm{Hom}_{\mathcal{T}(M)}(F_1, F_2[l]) \simeq H^l(M; k_M)$ for any $l \in \mathbb{Z}$.

Let \widehat{L} be a conic Lagrangian submanifold of $T^*(M \times \mathbb{R})$ obtained as a conification of some compact connected exact Lagrangian submanifold L of T^*M .

Lemma 4.1. *One has an inclusion*

$$\mathbf{D}_{\widehat{L} \cup T_{M \times \mathbb{R}}^*(M \times \mathbb{R}), +}^b(M \times \mathbb{R}) \subset {}^\perp \mathbf{D}_{\{\tau \leq 0\}}^b(M \times \mathbb{R}). \quad (4.1)$$

Proof. There exists $B \in \mathbb{R}$ such that $\widehat{L} \subset T^*(M \times (B, +\infty))$. Let $G \in \mathbf{D}_{\{\tau \leq 0\}}^b(M \times \mathbb{R})$. Since $\widehat{L} \subset \{\tau > 0\}$, by Proposition 2.8 we have

$$\mathrm{SS}(R\mathcal{H}om(F, G)) \subset \{\tau \leq 0\}. \quad (4.2)$$

Applying the microlocal Morse lemma (Proposition 2.3) to $R\mathcal{H}om(F, G)$ and the function $t: M \times \mathbb{R} \rightarrow \mathbb{R}, (x, t) \mapsto t$, we obtain the vanishing $R\mathcal{H}om(F, G) \simeq 0$ by the inclusion $\mathrm{Supp}(R\mathcal{H}om(F, G)) \subset M \times [B, +\infty)$. \square

Proposition 4.2. *There exists $c_0 \geq 0$ such that $\mathrm{Hom}_{\mathcal{D}(M)}(F_1, T_{c*}F_2[l])$ is isomorphic to $H^l(M; k_M)$ for any $c \geq c_0$ and $l \in \mathbb{Z}$. In particular, one has an isomorphism*

$$\mathrm{Hom}_{\mathcal{T}(M)}(F_1, F_2[l]) \simeq H^l(M; k_M) \quad \text{for any } l \in \mathbb{Z}. \quad (4.3)$$

Proof. By Lemma 4.1, for any $l \in \mathbb{Z}$ we have

$$\mathrm{Hom}_{\mathcal{D}(M)}(F_1, T_{c*}F_2[l]) = \mathrm{Hom}_{\mathbf{D}^b(M \times \mathbb{R})}(F_1, T_{c*}F_2[l]). \quad (4.4)$$

By the compactness of \widehat{L}_1 and \widehat{L}_2 , there exist $A_1, A_2 > 0$ satisfying $\widehat{L}_1 \subset T^*(M \times (-A_1, A_1))$ and $\widehat{L}_2 \subset T^*(M \times (-A_2, A_2))$. Take a sufficiently large $c_0 \geq 0$ such that $c_0 - A_2 > A_1$. Since $F_1|_{M \times (A_1, +\infty)} \simeq k_{M \times (A_1, +\infty)}$ and $\mathrm{Supp}(T_{c*}F_2) \subset M \times (c - A_2, +\infty)$, we get

$$R\mathcal{H}om(F_1, T_{c*}F_2) \simeq R\mathcal{H}om(k_{M \times \mathbb{R}}, T_{c*}F_2) \simeq R\Gamma(M \times \mathbb{R}; F_2) \quad (4.5)$$

for any $c \geq c_0$. Since $\mathrm{SS}(F_2) \subset \{\tau \geq 0\}$, we can apply the microlocal Morse lemma (Proposition 2.3) and obtain

$$\begin{aligned} R\Gamma(M \times \mathbb{R}; F_2) &\simeq R\Gamma(M \times (A_2, +\infty); F_2) \\ &\simeq R\Gamma(M \times (A_2, +\infty); k_{M \times (A_2, +\infty)}) \simeq R\Gamma(M; k_M). \end{aligned}$$

The second assertion follows from Proposition 3.8. \square

Remark 4.3. For the special case L is the zero-section T_M^*M of T^*M , the isomorphism $\mathrm{Hom}_{\mathcal{T}(M)}(F_{\widehat{L}}, F_{\widehat{L}}[l]) \simeq H^l(M; k_M)$ was already proved by Guillermou-Schapira [GS14]. In this case, $k_M \boxtimes k_{[0, +\infty)}$ is the canonical quantization associated with the zero-section T_M^*M . Guillermou-Schapira [GS14] proved that the functor

$$j_M: \mathbf{D}^b(M) \longrightarrow \mathcal{T}(M), \quad F \longmapsto F \boxtimes k_{[0, +\infty)} \quad (4.6)$$

is fully faithful (see [GS14, Corollary 5.8]). We thus obtain

$$\mathrm{Hom}_{\mathcal{T}(M)}(j_M(k_M), j_M(k_M)[l]) \simeq \mathrm{Hom}_{\mathbf{D}^b(M)}(k_M, k_M[l]) \simeq H^l(M; k_M). \quad (4.7)$$

In fact, we can prove (4.3) for general L_1 and L_2 from the above special case (4.7). The following was pointed out to the author by Tatsuki Kuwagaki.

Proposition 4.4. *For any compact connected exact Lagrangian submanifold L of T^*M , one has an isomorphism*

$$F_{\widehat{L}} \simeq j_M(k_M) \simeq k_{M \times [0, +\infty)} \quad \text{in } \mathcal{T}(M). \quad (4.8)$$

Proof. By the compactness of L , we can take a sufficiently large $A > 0$ such that $\widehat{L} \subset T^*(M \times (-A, A))$. Since $F_{\widehat{L}}|_{M \times (A, +\infty)} \simeq k_{M \times (A, +\infty)}$, there exists a canonical morphism

$$F_{\widehat{L}} \longrightarrow k_{M \times [A+1, +\infty)}. \quad (4.9)$$

The cone of this morphism is supported in $M \times [-A, A+1]$ and hence a torsion object. Therefore the morphism (4.9) is an isomorphism in $\mathcal{T}(M)$. A similar argument shows that the morphism $k_{M \times [A+1, +\infty)} \rightarrow k_{M \times [0, +\infty)}$ is an isomorphism in $\mathcal{T}(M)$. We thus obtain the result. \square

Hence (4.3) follows from the isomorphism (4.7).

Corollary 4.5. *Let L_1 and L_2 be compact connected exact Lagrangian submanifolds of T^*M . Then L_1 and L_2 are mutually non-displaceable.*

Proof. It follows from Theorem 3.10 and Proposition 4.2. \square

4.2 Lower bounds of the cardinality of intersection of two compact exact Lagrangians

Next we prove that the cardinality of the intersection $\#(L_1 \cap L_2)$ is bounded from below by the dimension of the Hom space between the canonical sheaf quantizations of L_1 and L_2 in the category $\mathcal{T}(M)$.

Hereafter we assume that

$$L_1 \text{ and } L_2 \text{ intersect transversally.} \quad (4.10)$$

Recall the isomorphism

$$\mathcal{H}om^*(F, G) \simeq Rs_* R\mathcal{H}om(q_2^{-1}i^{-1}F, q_1^!G). \quad (4.11)$$

Since q_2 and q_1 are submersions, by Theorem 2.6 (ii) we have

$$\begin{aligned} \mathring{SS}(q_2^{-1}i^{-1}F_1) &\subset q_{2d}q_{2\pi}^{-1}\mathring{SS}(i^{-1}F_1) \\ &= \{(x, t_1, t_2; \tau_1\xi_1, 0, -\tau_1) \mid \tau_1 > 0, (x; \xi_1) \in L_1, t_1 \in \mathbb{R}, t_2 = f_1(x; \xi_1)\} \end{aligned} \quad (4.12)$$

and

$$\begin{aligned} \mathring{SS}(q_1^!F_2) &\subset q_{1d}q_{1\pi}^{-1}\mathring{SS}(F_2) \\ &= \{(x, t_1, t_2; \tau_2\xi_2, \tau_2, 0) \mid \tau_2 > 0, (x; \xi_2) \in L_2, t_1 = -f_2(x; \xi_2), t_2 \in \mathbb{R}\}. \end{aligned} \quad (4.13)$$

Hence $\mathring{SS}(q_2^{-1}i^{-1}F_1) \cap \mathring{SS}(q_1^!F_2) = \emptyset$ and by Proposition 2.8, we obtain

$$\begin{aligned} \mathring{SS}(R\mathcal{H}om(q_2^{-1}i^{-1}F_1, q_1^!F_2)) &\subset \mathring{SS}(q_2^{-1}i^{-1}F_1)^a + \mathring{SS}(q_1^!F_2) \\ &= \left\{ (x, t_1, t_2; -\tau_1\xi_1 + \tau_2\xi_2, \tau_2, \tau_1) \left| \begin{array}{l} \tau_1, \tau_2 > 0, (x; \xi_1) \in L_1, (x; \xi_2) \in L_2, \\ t_2 = f_1(x; \xi_1), t_1 = -f_2(x; \xi_2) \end{array} \right. \right\} =: \Lambda_{M \times \mathbb{R} \times \mathbb{R}}. \end{aligned} \quad (4.14)$$

Set $H := R\mathcal{H}om(q_2^{-1}i^{-1}F_1, q_1^!F_2)$.

Proposition 4.6. *Let $\dot{p}' = (\dot{x}, \dot{t}_1, \dot{t}_2; 0, 1, 1) \in \Lambda_{M \times \mathbb{R} \times \mathbb{R}}$. Let moreover*

$$\{p_1, \dots, p_{N'}\} := \{p \in L_1 \cap L_2 \mid \dot{x} = \pi(p), \dot{t}_2 = f_1(p), \dot{t}_1 = -f_2(p)\}. \quad (4.15)$$

(Hence $N' = \#\{p \in L_1 \cap L_2 \mid \dot{x} = \pi(p), \dot{t}_2 = f_1(p), \dot{t}_1 = -f_2(p)\}$.)

(i) *For any $j = 1, \dots, N'$, there is a sufficiently small neighborhood W_j of p_j in T^*M such that*

$$\Lambda_j := \left\{ (x, t_1, t_2; -\tau_1 \xi_1 + \tau_2 \xi_2, \tau_2, \tau_1) \left| \begin{array}{l} \tau_1, \tau_2 > 0, \\ (x; \xi_1) \in L_1 \cap W_j, (x; \xi_2) \in L_2 \cap W_j, \\ t_2 = f_1(x; \xi_1), t_1 = -f_2(x; \xi_2) \end{array} \right. \right\} \quad (4.16)$$

is a smooth Lagrangian submanifold of $T^(M \times \mathbb{R} \times \mathbb{R})$.*

(ii) *There exist $H_j \in \mathbf{D}^b(M \times \mathbb{R} \times \mathbb{R})$ simple along Λ_j at \dot{p}' ($j = 1, \dots, N'$) such that*

$$H \simeq \bigoplus_{j=1}^{N'} H_j \quad \text{in } \mathbf{D}^b(M \times \mathbb{R} \times \mathbb{R}; \dot{p}'). \quad (4.17)$$

Proof. Consider the projections

$$\begin{aligned} \tilde{q}_1: M \times \mathbb{R} \times M \times \mathbb{R} &\longrightarrow M \times \mathbb{R}, \quad (x_1, t_1, x_2, t_2) \longmapsto (x_1, t_1), \\ \tilde{q}_2: M \times \mathbb{R} \times M \times \mathbb{R} &\longrightarrow M \times \mathbb{R}, \quad (x_1, t_1, x_2, t_2) \longmapsto (x_2, t_2) \end{aligned} \quad (4.18)$$

and the diagonal embedding

$$\delta: M \times \mathbb{R} \times \mathbb{R} \longrightarrow M \times \mathbb{R} \times M \times \mathbb{R}, \quad (x, t_1, t_2) \longmapsto (x, t_1, x, t_2). \quad (4.19)$$

Then we have an isomorphism

$$H \simeq \delta^! R\mathcal{H}om(\tilde{q}_2^{-1} i^{-1} F_1, \tilde{q}_2^! F_2). \quad (4.20)$$

By Proposition 2.7, we get

$$\mathrm{SS}(R\mathcal{H}om(\tilde{q}_2^{-1} i^{-1} F_1, \tilde{q}_2^! F_2)) \subset (\widehat{L}_2 \cup T_{M \times \mathbb{R}}^*(M \times \mathbb{R})) \times (\widehat{L}_1^{a'} \cup T_{M \times \mathbb{R}}^*(M \times \mathbb{R})). \quad (4.21)$$

Here the involution $a': T^*(M \times \mathbb{R}) \rightarrow T^*(M \times \mathbb{R})$ is defined by

$$a'(x, t; \xi, \tau) = (x, -t; -\xi, \tau). \quad (4.22)$$

Moreover $R\mathcal{H}om(\tilde{q}_2^{-1} i^{-1} F_1, \tilde{q}_2^! F_2)$ is simple along $\widehat{L}_2 \times \widehat{L}_1^{a'}$ by Proposition 2.17. Furthermore, by (4.21) we find that δ is non-characteristic for $\mathrm{SS}(R\mathcal{H}om(\tilde{q}_2^{-1} i^{-1} F_1, \tilde{q}_2^! F_2))$. Hence we obtain an isomorphism

$$\delta^! R\mathcal{H}om(\tilde{q}_2^{-1} i^{-1} F_1, \tilde{q}_2^! F_2) \simeq \omega_\delta \otimes \delta^{-1} R\mathcal{H}om(\tilde{q}_2^{-1} i^{-1} F_1, \tilde{q}_2^! F_2), \quad (4.23)$$

where $\omega_\delta = \omega_{M \times \mathbb{R} \times \mathbb{R} / M \times \mathbb{R} \times M \times \mathbb{R}}$ is the relative dualizing complex. By the assumption that L_1 and L_2 intersect transversally, δ_π is transversal to $\widehat{L}_2 \times \widehat{L}_1^{a'}$ at $((p_j, \dot{t}_1; 1), (p_j^a, \dot{t}_2; 1)) \in T^*(M \times \mathbb{R} \times M \times \mathbb{R})$ for any j . Thus (i) follows Proposition 2.18 (i), and (ii) follows from Proposition 2.13 (ii) and Proposition 2.18 (ii). \square

Lemma 4.7. *One has*

$$\begin{aligned} v_d^{-1} \left(\overline{v_\pi(\Lambda_{M \times \mathbb{R} \times \mathbb{R}} \cup T_{M \times \mathbb{R} \times \mathbb{R}}^*(M \times \mathbb{R} \times \mathbb{R}))} \right) &= v_d^{-1} v_\pi(\Lambda_{M \times \mathbb{R} \times \mathbb{R}} \cup T_{M \times \mathbb{R} \times \mathbb{R}}^*(M \times \mathbb{R} \times \mathbb{R})) \\ &= s_\pi s_d^{-1}(\Lambda_{M \times \mathbb{R} \times \mathbb{R}} \cup T_{M \times \mathbb{R} \times \mathbb{R}}^*(M \times \mathbb{R} \times \mathbb{R})). \end{aligned} \quad (4.24)$$

In other words,

$$s_\#(\Lambda_{M \times \mathbb{R} \times \mathbb{R}} \cup T_{M \times \mathbb{R} \times \mathbb{R}}^*(M \times \mathbb{R} \times \mathbb{R})) = s_\pi s_d^{-1}(\Lambda_{M \times \mathbb{R} \times \mathbb{R}} \cup T_{M \times \mathbb{R} \times \mathbb{R}}^*(M \times \mathbb{R} \times \mathbb{R})). \quad (4.25)$$

(See Subsection 2.4 for the notation v_π, v_d , and $s_\#$ associated with the constant linear map $s: M \times \mathbb{R} \times \mathbb{R} \rightarrow M \times \mathbb{R}$.)

Proof. Set

$$\Lambda := \left\{ (x, t; -\tau_1 \xi_1 + \tau_2 \xi_2, \tau_2, \tau_1) \mid \begin{array}{l} \tau_1, \tau_2 > 0, (x; \xi_1) \in L_1, (x; \xi_2) \in L_2, \\ t = f_1(x; \xi_1) - f_2(x; \xi_2) \end{array} \right\}. \quad (4.26)$$

Then $v_\pi(\Lambda_{M \times \mathbb{R} \times \mathbb{R}} \cup T_{M \times \mathbb{R} \times \mathbb{R}}^*(M \times \mathbb{R} \times \mathbb{R}))$ is equal to $\Lambda \cup (T_M^* M \times \mathbb{R} \times \{(0, 0)\}) \subset T^* M \times \mathbb{R} \times (\mathbb{R} \times \mathbb{R})$. For the first equality, it suffices to check the equality for $\tau_1 = \tau_2 = 0$. By the compactness of L_1 and L_2 , there exists $C > 0$ such that $|\xi| \leq C(|\tau_1| + |\tau_2|)$ for any $(x, t; \xi, \tau_1, \tau_2) \in \Lambda$. Therefore the same inequality holds on the closure $\bar{\Lambda}$ of Λ . Hence if $\tau_1 = \tau_2 = 0$ then $\xi = 0$. \square

By Proposition 2.10, Lemma 4.7, and (4.14), the set $\mathring{\text{SS}}(\mathcal{H}om^*(F_1, F_2)) = \mathring{\text{SS}}(Rs_* H)$ can be estimated as

$$\begin{aligned} \mathring{\text{SS}}(\mathcal{H}om^*(F_1, F_2)) &\subset s_\#(\Lambda_{M \times \mathbb{R} \times \mathbb{R}} \cup T_{M \times \mathbb{R} \times \mathbb{R}}^*(M \times \mathbb{R} \times \mathbb{R})) \cap \mathring{T}^*(M \times \mathbb{R}) \\ &= s_\pi s_d^{-1}(\Lambda_{M \times \mathbb{R} \times \mathbb{R}} \cup T_{M \times \mathbb{R} \times \mathbb{R}}^*(M \times \mathbb{R} \times \mathbb{R})) \cap \mathring{T}^*(M \times \mathbb{R}) \\ &\subset \left\{ (x, t; \tau(-\xi_1 + \xi_2), \tau) \mid \begin{array}{l} \tau > 0, (x; \xi_1) \in L_1, (x; \xi_2) \in L_2, \\ t = f_1(x; \xi_1) - f_2(x; \xi_2) \end{array} \right\} =: \Lambda_{M \times \mathbb{R}}. \end{aligned} \quad (4.27)$$

Hence we obtain

$$\begin{aligned} \Gamma_{dt} \cap \mathring{\text{SS}}(\mathcal{H}om^*(F_1, F_2)) &\subset \{(x, t; 0, 1)\} \cap \Lambda_{M \times \mathbb{R}} \\ &= \{(x, t; 0, 1) \mid \exists (x; \xi) \in L_1 \cap L_2, t = f_1(x; \xi) - f_2(x; \xi)\}. \end{aligned} \quad (4.28)$$

Proposition 4.8. *Let $p_0 = (x_0, t_0; 0, 1) \in \Lambda_{M \times \mathbb{R}}$. Then one has*

$$\begin{aligned} \sum_{l \in \mathbb{Z}} \dim H_{M \times [t_0, +\infty)}^l(\mathcal{H}om^*(F_1, F_2))_{(x_0, t_0)} \\ = \#\{p \in L_1 \cap L_2 \mid x_0 = \pi(p), t_0 = f_1(p) - f_2(p)\}. \end{aligned} \quad (4.29)$$

Proof. Take a relatively compact contractible open neighborhood U of x_0 and a sufficiently small $\varepsilon > 0$, and consider an open neighborhood $U_0 := U \times (t_0 - \varepsilon, t_0 + \varepsilon)$ of (x_0, t_0) in $M \times \mathbb{R}$. Then there exists a sufficiently large $A > 0$ such that F_1 is constant on $M \times (A - 1, +\infty)$ and $q_1^! F_2$ is constant on $s^{-1}(U_0) \cap (M \times \mathbb{R} \times (-\infty, -A + 1))$. Thus we have isomorphisms

$$\begin{aligned} R\mathcal{H}om(q_2^{-1} i^{-1} k_{M \times [A, +\infty)}, q_1^! F_2)|_{s^{-1}(U_0)} \\ \simeq R\mathcal{H}om(k_{M \times \mathbb{R} \times (-\infty, -A]}, k_{M \times \mathbb{R} \times \mathbb{R}}[1])|_{s^{-1}(U_0)} \\ \simeq R\Gamma_{s^{-1}(U_0) \cap (M \times \mathbb{R} \times (-\infty, -A])}(k_{s^{-1}(U_0)}[1]). \end{aligned} \quad (4.30)$$

Here the first isomorphism follows from Theorem 2.6 (ii), since the fibers of the submersion q_2 are one-dimensional and $s^{-1}(U_0)$ is contractible. Therefore we obtain

$$Rs_* R\mathcal{H}om(q_2^{-1}i^{-1}k_{M \times [A, +\infty)}, q_1^! F_2)|_{s^{-1}(U_0)} \simeq 0. \quad (4.31)$$

By the distinguished triangle

$$F'_1 \longrightarrow F_1 \longrightarrow k_{M \times [A, +\infty)} \xrightarrow{+1} \quad (4.32)$$

with F'_1 is supported in some compact subset, we find that

$$Rs_* R\mathcal{H}om(q_2^{-1}i^{-1}F_1, q_1^! F_2)|_{s^{-1}(U_0)} \simeq Rs_* R\mathcal{H}om(q_2^{-1}i^{-1}F'_1, q_1^! F_2)|_{s^{-1}(U_0)} \quad (4.33)$$

and s is proper on $\text{Supp}(R\mathcal{H}om(q_2^{-1}i^{-1}F'_1, q_1^! F_2)|_{s^{-1}(U_0)})$. Replacing F_1 by F'_1 , we may assume that s is proper on the support from the beginning.

The inclusion (4.14) shows that

$$\begin{aligned} s_\pi^{-1}(p_0) \cap s_d^{-1}(\text{SS}(H)) &\subset s_\pi^{-1}(p_0) \cap s_d^{-1}(\Lambda_{M \times \mathbb{R} \times \mathbb{R}}) \\ &= \left\{ (x_0, -f_2(x_0; \xi), f_1(x_0; \xi); 0, 1) \mid \begin{array}{l} \exists (x_0; \xi) \in L_1 \cap L_2, \\ t_0 = f_1(x_0; \xi) - f_2(x_0; \xi) \end{array} \right\} \\ &:= \{p_1, \dots, p_N\} \subset (M \times \mathbb{R} \times \mathbb{R}) \times_{(M \times \mathbb{R})} T^*(M \times \mathbb{R}), \end{aligned}$$

where N satisfies

$$N = \# \left\{ (t_1, t_2) \in \mathbb{R} \times \mathbb{R} \mid \begin{array}{l} t_1 + t_2 = t_0, \exists (x_0; \xi) \in L_1 \cap L_2, \\ t_1 = -f_2(x_0; \xi), t_2 = f_1(x_0; \xi) \end{array} \right\}. \quad (4.34)$$

At each point $p'_i := s_d(p_i) = (x_0, t_{1,i}, t_{2,i}; 0, 1, 1) \in \Lambda_{M \times \mathbb{R} \times \mathbb{R}}$, by Proposition 4.8 there exist smooth Lagrangian submanifolds $\Lambda_{i,j}$ of the forms as in (4.16) and $H_{i,j} \in \mathbf{D}^b(M \times \mathbb{R} \times \mathbb{R})$ simple along $\Lambda_{i,j}$ at p'_i ($j = 1, \dots, N'_i$) such that

$$H \simeq \bigoplus_{j=1}^{N'_i} H_{i,j} \quad \text{in } \mathbf{D}^b(M \times \mathbb{R} \times \mathbb{R}; p'_i) \quad (4.35)$$

and $N'_i = \#\{p \in L_1 \cap L_2 \mid x = \pi(p), t_{2,i} = f_1(p), t_{1,i} = -f_2(p)\}$. Moreover, by the transversality of L_1 and L_2 , s_d is transversal to $\Lambda_{i,j}$ at p'_i for any i and j . By Proposition 2.19 (i), for any $i = 1, \dots, N$, there exists a sufficiently small neighborhood W_i of p_i such that $s_\pi(s_d^{-1}(\Lambda_{i,j}) \cap W_i)$ ($j = 1, \dots, N'_i$) are smooth Lagrangian submanifolds of $T^*(M \times \mathbb{R})$. Furthermore, each $s_\pi(s_d^{-1}(\Lambda_{i,j}) \cap W_i)$ is of the form

$$\left\{ (x, t; \tau(-\xi_1 + \xi_2), \tau) \mid \begin{array}{l} \tau > 0, \\ (x; \xi_1) \in L_1 \cap W, (x; \xi_2) \in L_2 \cap W, \\ t = f_1(x; \xi_1) - f_2(x; \xi_2) \end{array} \right\}, \quad (4.36)$$

where W is a sufficiently small neighborhood of some point $(x_0; \xi_0) \in L_1 \cap L_2$ satisfying $t_0 = f_1(x_0; \xi_0) - f_2(x_0; \xi_0)$. Hence the transversality of L_1 and L_2 implies that the function $t: M \times \mathbb{R} \rightarrow \mathbb{R}, (x, t) \mapsto t$ is transversal to $s_\pi(s_d^{-1}(\Lambda_{i,j}) \cap W_i)$ for any i and j . Thus the result follows from Proposition 2.19 (ii). \square

Proposition 4.9. *Let $a < b$ in \mathbb{R} or $a \in \mathbb{R}, b = +\infty$. Assume that*

$$a \neq f_1(p) - f_2(p) \quad \text{for any } p \in L_1 \cap L_2. \quad (4.37)$$

Then one has an inequality

$$\begin{aligned} & \#\{p \in L_1 \cap L_2 \mid a < f_1(p) - f_2(p) < b\} \\ & \geq \sum_{l \in \mathbb{Z}} \dim H_{M \times [a, b]}^l(M \times (-\infty, b); \mathcal{H}om^*(F_1, F_2)). \end{aligned} \quad (4.38)$$

Proof. By (4.27) and (4.37), we have $\mathring{\text{SS}}(\mathcal{H}om^*(F_1, F_2)) \cap \mathring{\text{SS}}(k_{M \times [a, +\infty)}) = \emptyset$. Hence by Proposition 2.8 we obtain

$$\begin{aligned} \mathring{\text{SS}}(R\Gamma_{M \times [a, +\infty)} \mathcal{H}om^*(F_1, F_2)) &= \mathring{\text{SS}}(R\mathcal{H}om(k_{M \times [a, +\infty)}, \mathcal{H}om^*(F_1, F_2))) \\ &\subset \Lambda_{M \times \mathbb{R}} \cap \pi^{-1}(\{t > a\}) + \{(x, a; 0, -\tau') \mid \tau' > 0\}. \end{aligned} \quad (4.39)$$

Set $\tilde{H} := R\Gamma_{M \times [a, +\infty)}(\mathcal{H}om^*(F_1, F_2))|_{M \times (-\infty, b)} \in \mathbf{D}^b(M \times (-\infty, b))$. We shall apply the Morse inequality for sheaves (Theorem 2.4) to \tilde{H} and the function $t: M \times (-\infty, b) \rightarrow \mathbb{R}, (x, t) \mapsto t$. Combining (4.28) with (4.39), we get

$$\Gamma_{dt} \cap \text{SS}(\tilde{H}) \subset \{(x, t; 0, 1) \mid \exists (x; \xi) \in L_1 \cap L_2, a < t = f_1(x; \xi) - f_2(x; \xi) < b\} \quad (4.40)$$

and the intersection is finite. Moreover by Proposition 4.8, the inclusion is in fact an equality. Let $\{p'_1, \dots, p'_N\} := \Gamma_{dt} \cap \text{SS}(\tilde{H}), y_i = (x_i, t_i) := \pi(p'_i)$. We also set

$$W_i := R\Gamma_{M \times [t_i, b)}(\tilde{H})_{y_i}. \quad (4.41)$$

Then by Theorem 2.4, we get

$$\sum_{l \in \mathbb{Z}} b_l(R\Gamma(M \times (-\infty, b); \tilde{H})) \leq \sum_{l \in \mathbb{Z}} \sum_{i=1}^N b_l(W_i). \quad (4.42)$$

By Proposition 4.8, we obtain

$$\sum_{l \in \mathbb{Z}} b_l(W_i) = \#\{p \in L_1 \cap L_2 \mid x_i = \pi(p), t_i = f_1(p) - f_2(p)\}, \quad (4.43)$$

which shows

$$\sum_{l \in \mathbb{Z}} \sum_{i=1}^N b_l(W_i) = \#\{p \in L_1 \cap L_2 \mid a < f_1(p) - f_2(p) < b\}. \quad (4.44)$$

Combining (4.42) with (4.44), we have the conclusion. \square

Remark 4.10. C. Viterbo announced the relation between the sections of $\mathcal{H}om^*(F_1, F_2)$ on $M \times (-\infty, \lambda)$ and the Floer cochain complex $CF_{<\lambda}(L_1, L_2)$ filtered by $\{p \in L_1 \cap L_2 \mid f_1(p) - f_2(p) < \lambda\}$. Motivated by his work, we use sections not only on $M \times \mathbb{R}$ but also on $M \times (-\infty, b)$ in Proposition 4.9.

Theorem 4.11. *One has an inequality*

$$\#(L_1 \cap L_2) \geq \sum_{l \in \mathbb{Z}} \dim \text{Hom}_{\mathcal{T}(M)}(F_1, F_2[l]). \quad (4.45)$$

Proof. First recall the isomorphism in Proposition 3.8

$$\mathrm{Hom}_{\mathcal{T}(M)}(F_1, F_2[l]) \simeq \varinjlim_{c \rightarrow +\infty} \mathrm{Hom}_{\mathcal{D}(M)}(F_1, T_{c*}F_2[l]). \quad (4.46)$$

By Proposition 4.2, there exists a sufficiently large $c_0 \geq 0$ such that

$$\mathrm{Hom}_{\mathcal{T}(M)}(F_1, F_2[l]) \simeq \mathrm{Hom}_{\mathcal{D}(M)}(F_1, T_{c*}F_2[l]) \quad (4.47)$$

for any $c \geq c_0$ and $l \in \mathbb{Z}$. Moreover, replacing c_0 by a larger constant if necessary, we may assume that

$$\min_{p \in L_1 \cap L_2} (f_1(p) - f_2(p) + c_0) > 0. \quad (4.48)$$

By the uniqueness of the canonical quantization (Theorem 3.12), we have an isomorphism $T_{c*}F_{\widehat{L}_f} \simeq F_{\widehat{L}_{(f-c)}}$ ($c \in \mathbb{R}$). Hence, replacing f_2 by $f_2 - c_0$, we may assume that (4.47) holds for any $c \geq 0, l \in \mathbb{Z}$ and

$$\min_{p \in L_1 \cap L_2} (f_1(p) - f_2(p)) > 0 \quad (4.49)$$

from the beginning. Thus it suffices to show that

$$\#(L_1 \cap L_2) \geq \sum_{l \in \mathbb{Z}} \dim \mathrm{Hom}_{\mathcal{D}(M)}(F_1, F_2[l]). \quad (4.50)$$

Recall also the isomorphism in Proposition 3.3

$$\mathrm{Hom}_{\mathcal{D}(M)}(F_1, F_2[l]) \simeq H_{M \times [0, +\infty)}^l(M \times \mathbb{R}; \mathcal{H}om^*(F_1, F_2)). \quad (4.51)$$

Hence the result follows from Proposition 4.9 for the case $a = 0$ and $b = +\infty$. \square

Corollary 4.12 ([Nad09, Theorem 1.3.1] and [FSS08, Theorem 1]). *Let L_1 and L_2 be compact connected exact Lagrangian submanifolds of T^*M intersecting transversally. Then one has an inequality*

$$\#(L_1 \cap L_2) \geq \sum_{l \in \mathbb{Z}} \dim H^l(M; k_M). \quad (4.52)$$

Proof. It follows from Proposition 4.2 and Theorem 4.11. \square

A Functoriality of sheaf quantization

In this section, we prove the "functoriality" of Guillermou's canonical sheaf quantization with respect to Hamiltonian diffeomorphisms. We remark that results in this section are independent of the results in the previous sections and not used for the proofs of them.

Let L be a compact connected exact Lagrangian submanifold of T^*M and f a primitive of the Liouville form α . We define the conification \widehat{L}_f of L with respect to f as in (3.22). Let ψ be a Hamiltonian diffeomorphism of T^*M and $\phi = (\phi_s)_s: T^*M \times I \rightarrow T^*M$ be a Hamiltonian isotopy, where I is an open interval containing $[0, 1]$, such that $\phi_1 = \psi$ and $\phi_0 = \mathrm{id}_{T^*M}$. We denote by $H = (H_s)_s: T^*M \times I \rightarrow \mathbb{R}$ the associated Hamiltonian and by X_s the associated Hamiltonian vector field on T^*M . The homogeneous lift $\widehat{\phi}$ of ϕ is described as follows (see [GKS12, Proposition A.6]):

$$\widehat{\phi}_1(x, t; \xi, \tau) = (x', t + u(x; \xi/\tau); \xi', \tau), \quad (A.1)$$

where $(x'; \xi'/\tau) = \phi_1(x; \xi/\tau) = \psi(x; \xi/\tau)$ and

$$u = \int_0^1 (H_s - \alpha(X_s)) \circ \phi_s ds: T^*M \longrightarrow \mathbb{R}. \quad (\text{A.2})$$

Hence we get

$$\begin{aligned} \widehat{\phi}_1(\widehat{L}_f) &= \left\{ (x', t + u(x; \xi/\tau); \xi', \tau) \left| \begin{array}{l} \tau > 0, \exists (x; \xi) \text{ s.t. } (x; \xi/\tau) \in L, \\ (x'; \xi'/\tau) = \psi(x; \xi/\tau), t = -f(x; \xi/\tau) \end{array} \right. \right\} \\ &= \left\{ (x', t'; \xi', \tau) \left| \begin{array}{l} \tau > 0, (x'; \xi'/\tau) \in \psi(L), \\ t' = -f \circ \psi^{-1}(x'; \xi'/\tau) + u \circ \psi^{-1}(x'; \xi'/\tau) \end{array} \right. \right\}. \end{aligned}$$

On the other hand, we have equalities

$$\begin{aligned} \psi^* \alpha - \alpha &= \int_0^1 \left(\frac{d}{ds} \phi_s^* \alpha \right) ds \\ &= \int_0^1 \phi_s^* (L_{X_s} \alpha) ds \\ &= \int_0^1 \phi_s^* (d\iota_{X_s} \alpha + \iota_{X_s} d\alpha) ds \\ &= d \int_0^1 \phi_s^* (\alpha(X_s) - H_s) ds = -du. \end{aligned}$$

Here, for a vector field X , L_X denotes the Lie derivative with respect to X , and the third equality follows from Cartan's formula. Moreover, the fourth equality follows from the definition of the Hamiltonian vector field: $d\alpha(X_s, \cdot) = -dH_s$. Hence setting $\widetilde{f} := f \circ \psi^{-1} - u \circ \psi^{-1}: \psi(L) \rightarrow \mathbb{R}$, we get

$$\begin{aligned} \alpha|_{\psi(L)} &= (\psi^{-1})^* (\alpha|_L - du|_L) \\ &= (\psi^{-1})^* (df - du|_L) = d\widetilde{f}. \end{aligned} \quad (\text{A.3})$$

Thus we find that \widetilde{f} is a primitive of α on $\psi(L)$ and obtain the following.

Lemma A.1. *One has an equality*

$$\widehat{\phi}_1(\widehat{L}_f) = \widehat{\psi(L)}_{\widetilde{f}} \subset T^*(M \times \mathbb{R}). \quad (\text{A.4})$$

Proposition A.2. *Let $F_{\widehat{L}_f}$ be the canonical sheaf quantization of \widehat{L}_f . Let $\psi: T^*M \rightarrow T^*M$ be a Hamiltonian diffeomorphism and $\Psi: \mathbf{D}^b(M \times \mathbb{R}) \rightarrow \mathbf{D}^b(M \times \mathbb{R})$ the associated functor (see (3.18)). Then one has an isomorphism*

$$\Psi(F_{\widehat{L}_f}) \simeq F_{\widehat{\psi(L)}_{\widetilde{f}}} \quad \text{in } \mathbf{D}^b_{\widehat{\psi(L)}_{\widetilde{f}} \cup T^*_{M \times \mathbb{R}}(M \times \mathbb{R}), +}(M \times \mathbb{R}), \quad (\text{A.5})$$

where $\widetilde{f} := f \circ \psi^{-1} - u \circ \psi^{-1}: \psi(L) \rightarrow \mathbb{R}$.

Proof. By Lemma A.1, we have

$$\Psi(F_{\widehat{L}_f}) \in \mathbf{D}^b_{\widehat{\psi(L)}_{\widetilde{f}} \cup T^*_{M \times \mathbb{R}}(M \times \mathbb{R})}(M \times \mathbb{R}). \quad (\text{A.6})$$

By the uniqueness of the canonical quantization (Theorem 3.12), it remains to show that

$$\Psi(F_{\widehat{L}})_- \simeq 0, \quad \Psi(F_{\widehat{L}})_+ \simeq k_M. \quad (\text{A.7})$$

Let $\widehat{\phi}: \mathring{T}^*(M \times \mathbb{R}) \times I \rightarrow \mathring{T}^*(M \times \mathbb{R})$ be the associated homogeneous Hamiltonian isotopy and $K \in \mathbf{D}^{\text{lb}}(M \times \mathbb{R} \times M \times \mathbb{R} \times I)$ its sheaf quantization. Consider the composite $K \circ F_{\widehat{L}} \in \mathbf{D}^{\text{lb}}(M \times \mathbb{R} \times I)$. By the compactness of \widehat{L} , there exists $A > 0$ satisfying

$$\bigcup_{s \in [-\varepsilon, 1+\varepsilon]} \widehat{\phi}_s(\widehat{L}) \subset T^*(M \times (-A, A)). \quad (\text{A.8})$$

If necessary, shrinking I , we may assume that

$$\bigcup_{s \in I} \widehat{\phi}_s(\widehat{L}) \subset T^*(M \times (-A, A)) \quad (\text{A.9})$$

from the beginning. Set $G := (K \circ F_{\widehat{L}})|_{M \times (A, +\infty) \times I} \in \mathbf{D}^{\text{lb}}(M \times (A, +\infty) \times I)$. We shall show that

$$\text{SS}(G) \subset T_{M \times (A, +\infty) \times I}^*(M \times (A, +\infty) \times I). \quad (\text{A.10})$$

First by (2.18), we have

$$\text{SS}(K \circ F_{\widehat{L}}) \subset (\Lambda_{\widehat{\phi}} \circ \widehat{L}) \cup T_{M \times \mathbb{R} \times I}^*(M \times \mathbb{R} \times I). \quad (\text{A.11})$$

By the definition of $\Lambda_{\widehat{\phi}}$ (see (3.3)), we obtain

$$(\Lambda_{\widehat{\phi}} \circ \widehat{L}) \cap (T_{M \times \mathbb{R}}^*(M \times \mathbb{R}) \times T^*I) \subset T_{M \times \mathbb{R} \times I}^*(M \times \mathbb{R} \times I) \quad (\text{A.12})$$

Denote by $i_s: M \times \mathbb{R} \times \{s\} \hookrightarrow M \times \mathbb{R} \times I$ the closed embedding for any $s \in I$. Then by the definition of $\Lambda_{\widehat{\phi}}$, we also have

$$(i_s)_d(i_s)_\pi^{-1}(\Lambda_{\widehat{\phi}} \circ \widehat{L}) = \widehat{\phi}_s(\widehat{L}). \quad (\text{A.13})$$

Moreover by (A.9), we get

$$\widehat{\phi}_s(\widehat{L}) \cap T^*(M \times (A, +\infty)) = \emptyset \quad (\text{A.14})$$

for any $s \in I$. Hence the inclusion (A.10) follows from the above estimates (A.12), (A.13), and (A.14). Since I is contractible, we have an isomorphism

$$G \simeq p^{-1}(G|_{M \times (A, +\infty) \times \{0\}}), \quad (\text{A.15})$$

where $p: M \times (A, +\infty) \times I \rightarrow M \times (A, +\infty)$ is the projection. In particular, we get

$$\begin{aligned} \Psi(F_{\widehat{L}})|_{M \times (A, +\infty)} &= G|_{M \times (A, +\infty) \times \{1\}} \\ &\simeq G|_{M \times (A, +\infty) \times \{0\}} \\ &\simeq (F_{\widehat{L}})|_{M \times (A, +\infty)} \simeq k_{M \times (A, +\infty)} \end{aligned}$$

and $\Psi(F_{\widehat{L}})_+ \simeq k_M$. A similar argument shows that $\Psi(F_{\widehat{L}})_- \simeq 0$. \square

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